# A THEOREM ON SUMMABILITY FACTORS FOR THE WEIGHTED MEAN METHOD FOR DOUBLE SERIES IN ULTRAMETRIC FIELDS

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Abstract. Throughout this paper, $K$ denotes a complete, non-trivially valued, ultrametric (or non-archimedean) field. 4-dimensional infinite matrices, double sequences and double series have entries in $K$. In the present paper, we prove a theorem on summability factors for the Weighted mean method for double series in $K$.

**1 Introduction and Preliminaries** Throughout the present paper, $K$ denotes a complete, non-trivially valued, ultrametric (or non-archimedean) field. 4-dimensional infinite matrices, double sequences and double series have entries in $K$. We recall the following definitions and results briefly (for details, see [2]) for the sake of completeness.

**Definition 1.1.** *For a double sequence $\{x_{m,n}\}$ in $K$ and $x \in K$, we write*

$$^{1}\lim_{m+n\to\infty} x_{m,n} = x,$$

*if for every $\epsilon > 0$, the set $\{(m,n) \in \mathbb{N}^2 : |x_{m,n} - x| > \epsilon\}$ is finite, $\mathbb{N}$ being the set of non-negative integers. In such a case, $x$ is unique and $x$ is called the limit of the double sequence $\{x_{m,n}\}$. We also say that $\{x_{m,n}\}$ converges to $x$.*

**Definition 1.2.** *Let $\{x_{m,n}\}$ be a double sequence in $K$ and $s \in K$. We write*

$$\sum_{m,n=0}^{\infty,\infty} x_{m,n} = s,$$

*if*

$$\lim_{m+n\to\infty} s_{m,n} = s,$$

*where*

$$s_{m,n} = \sum_{k,\ell=0}^{m,n} x_{k,\ell}, \quad m,n = 0,1,2,\ldots.$$

*In such a case, we say that the double series $\sum_{m,n=0}^{\infty,\infty} x_{m,n}$ converges to $s$.*

**Remark 1.3.** *If $\{x_{m,n}\}$ converges, then $\{x_{m,n}\}$ is bounded.*

**Theorem 1.4.** *[2, Lemma 1] $\lim_{m+n\to\infty} x_{m,n} = x$ if and only if*

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*(i)* $\lim_{m\to\infty} x_{m,n} = x$, $n = 0, 1, 2, \ldots$;

*(ii)* $\lim_{n\to\infty} x_{m,n} = x$, $m = 0, 1, 2, \ldots$;
*and*

*(iii) for every* $\epsilon > 0$, *there exists* $N \in \mathbb{N}$ *such that* $|x_{m,n} - x| < \epsilon$, $m, n \geq N$, *which is written as*

$$\lim_{m,n\to\infty} x_{m,n} = x,$$

*noting that this is Pringsheim's definition of convergence of a double sequence.*

*Proof.* We can suppose that $x = 0$. Leaving out the trivial part of the theorem, let (i), (ii), (iii) hold. Using (iii), we can choose a positive integer $N_1$ such that

$$|x_{m,n}| \leq \epsilon, \quad m, n > N_1.$$

In view of (i), there exists a positive integer $N_2$ such that

$$|x_{m,n}| \leq \epsilon, \quad m > N_2, n = 0, 1, 2, \ldots, N_1.$$

In view of (ii), there exists a positive integer $N_3$ such that

$$|x_{m,n}| \leq \epsilon, \quad n > N_3, m = 0, 1, 2, \ldots, N_1.$$

Let $N = \max\{N_1, N_2, N_3\}$. Then it is possible that in the square $0 \leq m, n \leq N$,

$$|x_{m,n}| > \epsilon.$$

Note that outside this square,

$$|x_{m,n}| \leq \epsilon.$$

Thus,

$$\{(m,n) \in \mathbb{N}^2 : |x_{m,n}| > \epsilon\} \text{ is finite,}$$

$$i.e., \lim_{m+n\to\infty} x_{m,n} = 0,$$

completing the proof. □

**Theorem 1.5.** *[2, Lemma 2]* $\sum_{m,n=0}^{\infty,\infty} x_{m,n}$ *converges if and only if*

$$\lim_{m+n\to\infty} x_{m,n} = 0.$$

**Remark 1.6.** *Let* $K = \mathbb{Q}_2$, *the 2-adic field. Consider the series* $\sum_{m,n=0}^{\infty,\infty} x_{m,n}$, *where,* $x_{m,n} = 3^m 2^n$, $m, n = 0, 1, 2, \ldots$.

$$|x_{m,n}|_2 = |2|_2^n \to 0, \quad n \to \infty,$$

*since* $|3|_2 = 1$, *from which we have,*

$$\lim_{m,n\to\infty} x_{m,n} = 0.$$

*Easy calculation shows that*

$$S_{m,n} = \sum_{p,q=0}^{m,n} 3^p 2^q = \frac{(3^{m+1}-1)}{2}(2^{n+1}-1)$$

*and* $S_{m+1,n} - S_{m,n} = (2^{n+1}-1)3^{m+1}$,
*so that*

$$\begin{aligned}|S_{m+1,n} - S_{m,n}|_2 &= |2^{n+1}-1|_2 |3^{m+1}|_2 \\ &= 1.1 \\ &= 1 \not\to 0, \ m,n \to \infty,\end{aligned}$$

*using the fact that*

$$|a+b|_2 = \max(|a|_2, |b|_2),$$

*if* $|a|_2 \neq |b|_2$, $|\cdot|_2$ *being an ultrametric valuation. Consequently,* $\{S_{m,n}\}$ *is not Cauchy and so* $\sum_{m,n=0}^{\infty,\infty} 3^m 2^n$ *does not converge in the Pringsheim's sense. Thus* $\sum_{m,n=0}^{\infty,\infty} 3^m 2^n$ *diverges in the sense of Definition 1.1. Thus,* $\lim_{m,n\to\infty} x_{m,n} = 0$ *does not ensure convergence of* $\sum_{m,n=0}^{\infty,\infty} x_{m,n}$ *in the sense of Definition 1.1.*

**Remark 1.7.** *In the case of simple series, it is well-known that* $\sum_{n=0}^{\infty} x_n$ *converges if and only if*

$$\lim_{n\to\infty} x_n = 0.$$

*(see [1], p. 25, Theorem 1.1). Theorem 1.5 shows that Definition 1.1 is more suited in the ultrametric case than Pringsheim's definition of convergence of a double sequence.*

**Definition 1.8.** *Given a 4-dimensional infinite matrix* $A = (a_{m,n,k,\ell})$, $a_{m,n,k,\ell} \in K$, $m,n,k,\ell = 0,1,2,\ldots$ *and a double sequence* $\{x_{k,\ell}\}$, $x_{k,\ell} \in K$, $k,\ell = 0,1,2,\ldots$, *by the* $A$*-transform of* $x = \{x_{k,\ell}\}$, *we mean the double sequence* $A(x) = \{(Ax)_{m,n}\}$,

$$(Ax)_{m,n} = \sum_{k,\ell=0}^{\infty,\infty} a_{m,n,k,\ell} x_{k,\ell}, \quad m,n = 0,1,2,\ldots,$$

*where it is supposed that the double series on the right converge. If* $\lim_{m+n\to\infty} (Ax)_{m,n} = s$, *we say that the double sequence* $x = \{x_{k,\ell}\}$ *is* $A$*-summable or summable* $A$ *to* $s$, *written as*

$$x_{k,\ell} \to s(A).$$

*If* $\lim_{m+n\to\infty} (Ax)_{m,n} = s$, *whenever* $\lim_{k+\ell\to\infty} x_{k,\ell} = s$, *we say that* $A$ *is regular. A double series* $\sum_{m,n=0}^{\infty,\infty} x_{m,n}$ *is said to be* $A$*-summable to* $s$, *if* $\{s_{m,n}\}$ *is* $A$*-summable to* $s$, *where* $s_{m,n} = \sum_{k,\ell=0}^{m,n} x_{k,\ell}$, $m,n = 0,1,2,\ldots$.

The following important result, due to Natarajan and Srinivasan [2], gives a criterion for a 4-dimensional infinite matrix to be regular in terms of its entries.

**Theorem 1.9** (Silverman-Toeplitz). *The 4-dimensional infinite matrix $A = (a_{m,n,k,\ell})$ is regular if and only if*

$$\sup_{m,n,k,\ell} |a_{m,n,k,\ell}| < \infty; \tag{1}$$

$$\lim_{m+n\to\infty} a_{m,n,k,\ell} = 0, \quad k, \ell = 0, 1, 2, \dots; \tag{2}$$

$$\lim_{m+n\to\infty} \sum_{k,\ell=0}^{\infty,\infty} a_{m,n,k,\ell} = 1; \tag{3}$$

$$\lim_{m+n\to\infty} \sup_{k\geq 0} |a_{m,n,k,\ell}| = 0, \quad \ell = 0, 1, 2, \dots; \tag{4}$$

*and*

$$\lim_{m+n\to\infty} \sup_{\ell\geq 0} |a_{m,n,k,\ell}| = 0, \quad k = 0, 1, 2, \dots. \tag{5}$$

## 2 Weighted Mean Method for Double Sequences in $K$

The Weighted mean method $(\overline{N}, p_{m,n})$ for double sequences and double series in $K$ was introduced earlier by Natarajan and Sakthivel in [5].

**Definition 2.1.** *Given $p_{m,n} \in K$, $m, n = 0, 1, 2, \dots$, the Weighted mean method, denoted by $(\overline{N}, p_{m,n})$, is defined by the 4-dimensional infinite matrix $(a_{m,n,k,\ell})$, $m, n, k, \ell = 0, 1, 2, \dots$, where*

$$a_{m,n,k,\ell} = \begin{cases} \frac{p_{k,\ell}}{P_{m,n}}, & \text{if } k \leq m \text{ and } \ell \leq n; \\ 0, & \text{otherwise,} \end{cases}$$

$P_{m,n} = \sum_{k,\ell=0}^{m,n} p_{k,\ell}$, $m, n = 0, 1, 2, \dots$ *with the double sequence $\{p_{m,n}\}$ of weights satisfying the conditions:*

$$p_{m,n} \neq 0, \ m, n = 0, 1, 2, \dots;$$

*and for every fixed pair $(i, j)$,*

$$|p_{k,\ell}| \leq |P_{i,j}|, \quad \begin{matrix} k = 0, 1, 2, \dots, i; i = 0, 1, 2, \dots; \\ \ell = 0, 1, 2, \dots, j; j = 0, 1, 2, \dots. \end{matrix}$$

Natarajan and Sakthivel [5] proved the following result.

**Theorem 2.2.** *[5, Theorem 3.1] $(\overline{N}, p_{m,n})$ is regular if and only if*

$$\lim_{m+n\to\infty} |P_{m,n}| = \infty;$$

$$\lim_{m+n\to\infty} \frac{\max_{0\leq k\leq m} |p_{k,\ell}|}{P_{m,n}} = 0, \ \ell = 0, 1, 2, \dots;$$

*and*

$$\lim_{m+n\to\infty} \frac{\max_{0\leq \ell\leq n} |p_{k,\ell}|}{P_{m,n}} = 0, \ k = 0, 1, 2, \dots.$$

**3 Main Theorem** Some properties of the Weighted mean method for double sequences in $K$ were studied in [5].

A theorem on summability factors for the Weighted mean method for simple series in $K$ was proved in [4] and more generally, a theorem on summability factors for any regular method for simple series in $K$ was proved in [3]. For the definition of summability factors for simple series in the classical case, the reader can refer to [6], pp. 38-39. We retain the same definition for double series in the ultrametric set up with suitable changes.

We now prove the main result of the paper, which deals with summability factors for the Weighted mean method for double series in $K$.

**Theorem 3.1.** *If* $\sum_{m,n=0}^{\infty,\infty} a_{m,n}$ *is* $(\overline{N}, p_{m,n})$ *summable,* $(\overline{N}, p_{m,n})$ *being regular and if* $\{b_{m,n}\}$ *converges, then* $\sum_{m,n=0}^{\infty,\infty} a_{m,n}b_{m,n}$ *is* $(\overline{N}, p_{m,n})$ *summable too.*

*Proof.* Let $s_{m,n} = \sum_{k,\ell=0}^{m,n} a_{k,\ell}$, $t_{m,n} = \sum_{k,\ell=0}^{m,n} a_{k,\ell}b_{k,\ell}$, $m, n = 0, 1, 2, \ldots$. Let $\{\alpha_{m,n}\}$, $\{\beta_{m,n}\}$ be the $(\overline{N}, p_{m,n})$-transforms of $\{s_{m,n}\}$, $\{t_{m,n}\}$ respectively so that

$$\alpha_{m,n} = \frac{1}{P_{m,n}} \sum_{k,\ell=0}^{m,n} p_{k,\ell}s_{k,\ell},$$

$$\beta_{m,n} = \frac{1}{P_{m,n}} \sum_{k,\ell=0}^{m,n} p_{k,\ell}t_{k,\ell},$$

$m, n = 0, 1, 2, \ldots$. Let $\lim_{m+n\to\infty} \alpha_{m,n} = s$ and $\lim_{m+n\to\infty} b_{m,n} = m$. Let

$$b_{m,n} = m + \varepsilon_{m,n}, \ \ m, n = 0, 1, 2, \ldots$$

so that

$$\lim_{m+n\to\infty} \varepsilon_{m,n} = 0.$$

Now,

$$\begin{aligned}
\alpha_{m,n} &= \frac{1}{P_{m,n}} \sum_{k,\ell=0}^{m,n} p_{k,\ell}s_{k,\ell} \\
&= \frac{1}{P_{m,n}} \sum_{k,\ell=0}^{m,n} p_{k,\ell} \left( \sum_{i,j=0}^{k,\ell} a_{i,j} \right) \\
&= \frac{1}{P_{m,n}} \sum_{k,\ell=0}^{m,n} a_{k,\ell} \left( \sum_{i,j=k,\ell}^{k,\ell} p_{i,j} \right).
\end{aligned}$$

Similarly,

$$\beta_{m,n} = \frac{1}{P_{m,n}} \sum_{k,\ell=0}^{m,n} a_{k,\ell}b_{k,\ell} \left( \sum_{i,j=k,\ell}^{m,n} p_{i,j} \right).$$

Thus,

$$\begin{aligned}\beta_{m,n} &= \frac{1}{P_{m,n}} \sum_{k,\ell=0}^{m,n} a_{k,\ell}(m+\varepsilon_{k,\ell}) \left( \sum_{i,j=k,\ell}^{m,n} p_{i,j} \right) \\ &= m \left[ \frac{1}{P_{m,n}} \sum_{k,\ell=0}^{m,n} a_{k,\ell} \left( \sum_{i,j=k,\ell}^{m,n} p_{i,j} \right) \right] \\ &\quad + \frac{1}{P_{m,n}} \sum_{k,\ell=0}^{m,n} a_{k,\ell}\varepsilon_{k,\ell} \left( \sum_{i,j=k,\ell}^{m,n} p_{i,j} \right) \\ &= m\alpha_{m,n} + \frac{1}{P_{m,n}} \sum_{k,\ell=0}^{m,n} a_{k,\ell}\varepsilon_{k,\ell} \left( \sum_{i,j=k,\ell}^{m,n} p_{i,j} \right) \\ &= m\alpha_{m,n} + \sum_{k,\ell=0}^{\infty,\infty} a_{m,n,k,\ell}\varepsilon_{k,\ell},\end{aligned}$$

where the 4-dimensional infinite matrix $(a_{m,n,k,\ell})$, $m, n, k, \ell = 0, 1, 2, \ldots$ is defined by

$$a_{m,n,k,\ell} = \begin{cases} \dfrac{a_{k,\ell} \left( \sum_{i,j=k,\ell}^{m,n} p_{i,j} \right)}{P_{m,n}}, & \text{if } k \le m \text{ and } \ell \le n; \\ 0, & \text{otherwise.} \end{cases}$$

Using the fact that $(\overline{N}, p_{m,n})$ is regular, one can prove that $A$ transforms all null double sequences, i.e., all double sequences converging to zero into convergent double sequences.

Since $\lim_{k+\ell \to \infty} \varepsilon_{k,\ell} = 0$,

$$\lim_{m+n\to\infty} \left( \sum_{k,\ell=0}^{\infty,\infty} a_{m,n,k,\ell}\varepsilon_{k,\ell} \right) \text{ exists.}$$

Thus,

$$\lim_{m+n\to\infty} \beta_{m,n} = ms + \lim_{m+n\to\infty} \left( \sum_{k,\ell=0}^{\infty,\infty} a_{m,n,k,\ell}\varepsilon_{k,\ell} \right).$$

In other words, $\sum_{m,n=0}^{\infty,\infty} a_{m,n}b_{m,n}$ is $(\overline{N}, p_{m,n})$ summable to

$ms + \lim_{m+n\to\infty} \left( \sum_{k,\ell=0}^{\infty,\infty} a_{m,n,k,\ell}\varepsilon_{k,\ell} \right)$. This completes the proof of the theorem. □

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