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ON APPROXIMATE SOLUTION TO THE INVERSE QUASI-VARIATIONAL INEQUALITY PROBLEM

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Abstract. In the recent past, several existence theorems for the solution of inverse variational problem which is a special case of variational inequality problems have been established by several authors. In this paper, we have define an approximate solution to inverse quasi-variational inequality problem in a locally convex Hausdorff topological vector space.

1 Introduction The theory of variational inequality problems (VIP) and its applications are well known in the last five decades. The notion of inverse variational inequality problem (IVIP) has received the attention of researchers recently due to its applications in various fields, such as traffic network problems, economic equilibrium problems (see, for example [1]). Though, the inverse variational inequality problem is a special case (see [13]) of variational inequality problems, various authors [1, 11] have explored new sufficient conditions for the existence of solution to inverse variational inequality problem, because of the fact that the existence theorems for inverse variational inequality problem are stronger than those for variational inequality problems.

He et. al. [6] introduced the inverse variational inequality problem to study the bipartite market equilibrium problem. Zou et. al. [13] gave a novel method to solve inverse variational inequality problems based on neural networks.

Recently, Aussel et. al. [2] have studied the inverse quasi-variational inequality problem (IQVIP) with an application to road pricing problem and Han et. al. [1] have established the existence of solution to the inverse quasi-variational inequality problem using fixed point theorem and Fan-Knaster-Kuratowski-Mazurkiewicz (KKM) Lemma.

Let K be a non-empty subset of $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$. Let $\Phi : \mathbb{R}^n \to 2^K$ be a set-valued mapping. The inverse quasi-variational inequality problem is to find a vector $x \in \mathbb{R}^n$ such that

$$f(x) \in \Phi(x), \qquad \langle x, y - f(x)\rangle \geq 0, \forall y \in \Phi(x). \tag{1}$$

When $\Phi(x) = \Omega$ for all $x \in \mathbb{R}^n$, where Ω is a non-empty subset of $\mathbb{R}^n$, the inverse quasi-variational inequality problem reduces to the inverse variational inequality problem, that is, to find an $x \in \mathbb{R}^n$ such that

$$f(x) \in \Omega, \qquad \langle x, y - f(x)\rangle \geq 0, \forall y \in \Omega.$$

For more details, one can also refer to [3, 4, 5, 7, 8, 9, 10].

Han et. al. [1] proved the following existence theorem.

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Theorem 1.1 *Let $f^{-1}(K)$ be bounded convex and $K \subseteq f(\mathbb{R}^n)$ be compact. Assume that*
(i) f is continuous on $\mathbb{R}^n$ and natural quasi $\mathbb{R}^n_+$ -convex on $f^{-1}(K)$,
(ii) f is monotone on $f^{-1}(K)$, and $f^{-1}(.)$ is l.s.c. on K,
(iii) Φ is continuous on $\mathbb{R}^n$ and for each $u \in \mathbb{R}^n$, $\Phi(u)$ is convex closed, and $f^{-1}(\Phi(u))$ is bounded and convex with $f^{-1}(\Phi(u)) \subseteq \mathbb{R}^n_+$.
Then, the inverse quasi-variational inequality has a solution.

When there is no solution to the inverse quasi-variational inequality problem especially when the ranges of f and Φ do not intersect, one can look for an approximate solution, as there is no possibility of the existence of solution. In this paper we give sufficient conditions for the existence of an approximate solution to the inverse quasi-variational inequality problem in infinite dimensional setting.

2 Basic definitions and results Let K be a non-empty subset of a locally convex Hausdorff topological vector space X, p be a continuous semi-norm on X and $\langle \cdot, \cdot \rangle$ be a continuous bilinear functional on $X \times X$. Then, for any $x \in X$ define $d_p(x, K) = \inf \{p(x - y), \forall y \in K\}$. A point $z \in K$ is said to be a best approximation to x with respect to p from K if $p(z - x) = d_p(x, K)$. It is well known that[12] if K is a non-empty compact convex subset, then such a best approximation from K exists to given any x in X. We say that K is relatively compact if it's closure is compact.

Definition 2.1 *[15] Let X and Y be two Hausdorff topological spaces. A set-valued mapping $\Phi : X \to 2^Y$ is said to be*
(i) upper semi-continuous (in short u.s.c) at $x_0 \in X$ if for any neighbourhood $\mathcal{N}_0$ of $\Phi(x_0)$, there exists a neighbourhood $\mathcal{N}(x_0)$ of x_0 such that

$$\Phi(x) \subseteq \mathcal{N}_0, \textit{ for all } x \in \mathcal{N}(x_0).$$

(ii) lower semi-continuous (in short l.s.c) at $x_0 \in X$ if for any $y_0 \in \Phi(x_0)$ and any neighbourhood $\mathcal{N}(y_0)$ of y_0, there exists a neighbourhood $\mathcal{N}(x_0)$ of x_0 such that

$$\Phi(x) \cap \mathcal{N}(y_0) \neq \emptyset, \textit{ for all } x \in \mathcal{N}(x_0).$$

A set-valued mapping $\Phi : X \to 2^Y$ is said to be continuous at a point $x_0 \in X$ if it is both u.s.c and l.s.c at $x_0 \in X$. It is said to be continuous on X, if it is continuous at every point $x \in X$.

Definition 2.2 *[17] Let X and Y be two topological vector spaces. A set-valued mapping $\Phi : X \to 2^Y$ is said to be concave if*

$$\Phi(\lambda x + (1 - \lambda)y) \subseteq \lambda\Phi(x) + (1 - \lambda)\Phi(y), \textit{ for all } x, y \in X \textit{ and } \lambda \in [0, 1].$$

Lemma 2.3 *[15] Assume that X and Y are any two topological spaces and $\Phi : X \to 2^Y$ is a set-valued mapping. Then Φ is lower semi-continuous at $x_0 \in X$ if and only if for any net $\{x_\alpha\} \subseteq X$ with $x_\alpha \longrightarrow x_0$ and for any $y_0 \in \Phi(x_0)$, there exists a subnet $\{x_\beta\}$ of $\{x_\alpha\}$ and a net $y_\beta \in \Phi(x_\beta)$ such that $y_\beta \longrightarrow y_0$.*

Let K be a non-empty subset of X. We call a set-valued mapping $\Phi : K \to 2^K$ Kakutani factorizable [14] if $\Phi = \Phi_n \circ \Phi_{n-1} \circ ..... \circ \Phi_0$, that is, if there is a diagram

$$\Phi : K \xrightarrow{\Phi_0} K_1 \xrightarrow{\Phi_1} K_2 \to .... \xrightarrow{\Phi_n} K_{n+1} = K,$$

where for each Φ_i is a non-empty set-valued mapping and K_i is a convex subset of X. For such Kakutani factorizable mappings, Lassonde [14] proved the following fixed point theorem.

Theorem 2.4 *[14] Let K be non-empty convex subset of a locally convex Hausdorff topological vector space X and a set-valued mapping $\Phi : K \to 2^K$ be Kakutani factorizable, that is $\Phi{=}\Phi_n \circ \Phi_{n-1} \circ \ldots.. \circ \Phi_0$, where each Φ_i is non-empty compact convex valued upper semi-continuous set-valued mapping. If $\Phi(K)$ is relatively compact, then Φ has a fixed point, that is, there exists $x_0 \in K$ such that $x_0 \in \Phi(x_0)$.*

We end this section with the following theorem which we will need in the proof of our main theorem.

Theorem 2.5 *[12, Theorem B] Let E and F be two locally convex topological vector spaces, X be a non-empty compact and convex subset of E, Y be a non-empty subset of F, and $f, g : X \times Y \to \mathbb{R}$. If*
(i) $f(x, y) \leq g(x, y)$,
(ii) for each $x \in X$, $\{y \in Y : f(x, y) > 0\}$ is convex,
(iii) for each $y \in Y$, $x \to f(x, y)$ is lower semi-continuous on X,
(iv) for each $y \in Y$, $\{x \in X : g(x, y) \leq 0\}$ be non-empty and convex,
(v) g is lower semi-continuous on $X \times Y$,
then there exists $x_0 \in X$ such that $f(x_0, y) \leq 0$ for all $y \in Y$.

3 Existence of approximate solution to IQVIP

We now prove our main theorem.

Theorem 3.1 *Let K be a non-empty compact and convex subset of a locally convex Hausdorff topological vector space X. Let $f : K \to X$ be a continuous mapping and $\Phi : K \to 2^K$ be a continuous set-valued mapping with non-empty compact convex values, satisfying the following conditions:*
(i) Φ is concave and $\Phi(K)$ is convex
(ii) for x_1, $x_2 \in K$ and $u_1 \in \Phi(x_1)$, $u_2 \in \Phi(x_2)$, we have

$$\langle x_1, u_2 - z\rangle + \langle x_2, u_1 - z\rangle \geq 0, \text{ for all } z \in \mathbb{A}_p$$

(iii) for each $y \in \Phi(K)$, $\{x \in K : \langle x, z - y\rangle \leq 0\}$ is non-empty and convex for all $z \in \mathbb{A}_p$, where $\mathbb{A}_p = \bigcup_{x \in K} \{z \in \Phi(x) : p(z - f(x)) = d_p(f(x), \Phi(x))\}$.
Then, the inverse quasi-variational inequality problem (1) admits an approximate solution, that is, there exist $x_0 \in K$ and $z_0 \in \Phi(x_0)$ such that

$$p(z_0 - f(x_0)) = d_p(f(x_0), \Phi(x_0)) \text{ and } \langle x_0, y - z_0\rangle \geq 0, \text{ for all } y \in \Phi(x_0).$$

Proof. Define a set-valued mapping $S : K \to 2^K$ by S=$S_1 \circ S_0$, where $S_0 : K \to 2^{\mathbb{A}_p}$ and $S_1 : \mathbb{A}_p \to 2^K$ with

$$S_0(x) = \{z \in \Phi(x) : p(z - f(x)) = d_p(f(x), \Phi(x))\} \text{ and}$$
$$S_1(z) = \{\omega \in K : \langle \omega, y - z\rangle \geq 0, \forall y \in \Phi(\omega)\}.$$

We first claim that S is a Kakutani factorizable set-valued mapping. By our assumption, $\Phi(x)$ is compact, convex for each $x \in K$, and thus for every $x \in K$, $f(x)$ has a best approximation from $\Phi(x)$. Hence $S_0(x)$ is non-empty.

To show that $S_0(x)$ is closed, let $\{z_\alpha\}$ be any net in $S_0(x)$ which converges to z. We show that $z \in S_0(x)$. Since $\{z_\alpha\}$ belongs to $S_0(x)$,

$$p(z_\alpha - f(x)) = d_p(f(x), \Phi(x)).$$

As $\Phi(x)$ is compact, $z \in \Phi(x)$. Letting $\alpha \longrightarrow \infty$, we see that $d_p(f(x), \Phi(x)) = p(z - f(x))$, that is, $z \in S_0(x)$ and hence $S_0(x)$ is closed. For each $x \in K$, $\Phi(x)$ is compact, hence $S_0(x)$

is compact.

Also, $S_0(x)$ is convex for each $x \in K$. Indeed, let z_1 and z_2 belong to $S_0(x)$ for fixed $x \in K$. This implies that

$$p(z_1 - f(x)) = d_p(f(x), \Phi(x)) \text{ and } p(z_2 - f(x)) = d_p(f(x), \Phi(x)).$$

We show that for any $\lambda \in [0,1]$, $\quad \lambda z_1 + (1-\lambda)z_2 \in S_0(x)$. Since $\Phi(x)$ is convex, $\lambda z_1 + (1-\lambda)z_2 \in \Phi(x)$ for any $\lambda \in [0,1]$. For $\lambda \in [0,1]$

$$\begin{aligned} p(\lambda z_1 + (1-\lambda)z_2 - f(x)) &\leq \lambda p(z_1 - f(x)) + (1-\lambda)p(z_2 - f(x)) \\ &= \lambda d_p(f(x), \Phi(x)) + (1-\lambda)d_p(f(x), \Phi(x)) \\ &= d_p(f(x), \Phi(x)) \\ &\leq p(\lambda z_1 + (1-\lambda)z_2 - f(x)), \end{aligned}$$

which implies that $\lambda z_1 + (1-\lambda)z_2 \in S_0(x)$. Hence $S_0(x)$ is convex.

We now show that S_0 is upper semi-continuous. Let B be any non-empty closed subset of $\Phi(K)$. To show that $S_0^{-1}(B)$ is closed, it is enough to show that if $\omega_\alpha \in S_0^{-1}(B)$ and $\omega_\alpha \longrightarrow \omega$, then $\omega \in S_0^{-1}(B)$. Let $\omega_\alpha \in S_0^{-1}(B)$ and $\omega_\alpha \longrightarrow \omega$. This implies that $S_0(\omega_\alpha) \cap B \neq \emptyset$. Let $\zeta_\alpha \in S_0(\omega_\alpha) \cap B$. Since $\Phi(K)$ is compact, without loss of generality, we can assume that $\zeta_\alpha \longrightarrow \zeta$. This implies that $\zeta \in B$ as B is closed. Now we show that $\zeta \in S_0(\omega)$. Indeed, since $\zeta_\alpha \in S_0(\omega_\alpha)$, $p(\zeta_\alpha - f(\omega_\alpha)) = d_p(f(\omega_\alpha), \Phi(\omega_\alpha))$. Now, as $\alpha \longrightarrow \infty$, we get $p(\zeta - f(\omega)) = d_p(f(\omega), \Phi(\omega))$, that is, $\zeta \in S_0(\omega)$ and hence $\zeta \in S_0(\omega) \cap B$. Thus $S_0^{-1}(B)$ is closed and S_0 is upper semi-continuous.

We next show that $S_1(z)$ is non-empty. Fix $z \in \mathbb{A}_p$ and define $f_z : K \times \Phi(K) \to \mathbb{R}$ by $f_z(x,y) = \langle x, z - y \rangle$. By assumption (iii), the continuity of $\langle \cdot, \cdot \rangle$, it is easy to see that all the conditions of Theorem 2.5 are satisfied by taking $f_z(,) = g_z(,)$. Therefore there exists $x_0 \in K$ such that $\langle x_0, z - y \rangle \leq 0$ for all $y \in \Phi(K)$. In particular there exists $x_0 \in K$ such that $\langle x_0, z - y \rangle \leq 0$ for all $y \in \Phi(x_0)$, that is, there exists $x_0 \in K$ such that $\langle x_0, y - z \rangle \geq 0$ for all $y \in \Phi(x_0)$. Hence $S_1(z)$ is non-empty.

To show the compactness of $S_1(z)$, it is enough to show that it is closed. Let $\{x_\alpha\}$ be a net in $S_1(z)$ such that $x_\alpha \longrightarrow x$. Since $x_\alpha \in S_1(z)$, $\langle x_\alpha, y - z \rangle \geq 0$, for all $y \in \Phi(x_\alpha)$, for each α. Let us show that $\langle x, y - z \rangle \geq 0$, for all $y \in \Phi(x)$. Let $y \in \Phi(x)$. Since $x_\alpha \longrightarrow x$ and Φ is lower semi continuous, by Lemma 2.3, there exists a net $y'_\alpha \in \Phi(x_\alpha)$ such that $y'_\alpha \longrightarrow y$. This implies that $\langle x_\alpha, y'_\alpha - z \rangle \geq 0$, as $x_\alpha \in S_1(z)$. Since $x_\alpha \longrightarrow x$, $\langle \cdot, \cdot \rangle$ is continuous and $y'_\alpha \longrightarrow y$, as $\alpha \longrightarrow \infty$, we see that $\langle x, y - z \rangle \geq 0$. Since y is arbitrary, $\langle x, y - z \rangle \geq 0$, for all $y \in \Phi(x)$, that is, $x \in S_1(z)$ and hence $S_1(z)$ is closed. Since K is compact, $S_1(z)$ is compact.

Let us now show that $S_1(z)$ is convex. Let $p, q \in S_1(z)$ and $\lambda \in [0,1]$. That is, $\langle p, y - z \rangle \geq 0$, for all $y \in \Phi(p)$ and $\langle q, y' - z \rangle \geq 0$, for all $y' \in \Phi(q)$. It is enough to show that $\langle \lambda p + (1-\lambda)q, y - z \rangle \geq 0$, for all $y \in \Phi(\lambda p + (1-\lambda)q)$. Let $y \in \Phi(\lambda p + (1-\lambda)q)$. Since Φ is concave, we have $y = \lambda y_1 + (1-\lambda)y_2$, for some $y_1 \in \Phi(p), y_2 \in \Phi(q)$.

Now,

$$\begin{aligned}
&\langle \lambda p-(1-\lambda)q, y-z\rangle \\
&= \langle \lambda p-(1-\lambda)q, \lambda y_1+(1-\lambda)y_2-z\rangle \\
&= \lambda^2 \langle p, y_1-z\rangle + (1-\lambda)^2 \langle q, y_2-z\rangle + \lambda(1-\lambda)[\langle p, y_2-z\rangle + \langle q, y_1-z\rangle] \\
&\geq 0 \qquad\quad [\ p,q\in S_1(z) \ \textit{and assumption (ii)}],
\end{aligned}$$

which implies that $\langle \lambda p+(1-\lambda)q, y-z\rangle \geq 0$, for all $y\in\Phi(\lambda p+(1-\lambda)q)$. Thus $\lambda p+(1-\lambda)q\in S_1(z)$, for all $\lambda\in[0,1]$ and hence $S_1(z)$ is convex.

We now show that S_1 is upper semi-continuous. Let $B\subseteq K$ be closed and $\{z_\alpha\}$ be a net with $z_\alpha\in S_1^{-1}(B)$ such that $z_\alpha\longrightarrow z$ as $\alpha\longrightarrow\infty$. This implies that $S_1(z_\alpha)\cap B\neq\emptyset$, for all α. Let $y_\alpha\in S_1(z_\alpha)\cap B$ and $y_\alpha\longrightarrow y_0$ as $\alpha\longrightarrow\infty$. Since B is closed, $y_0\in B$. We have to show that $y_0\in S_1(z)$. Since $y_\alpha\in S_1(z_\alpha)$, $\langle y_\alpha, y-z_\alpha\rangle\geq 0$, for all $y\in\Phi(y_\alpha)$ and for all α. Let $y\in\Phi(y_0)$. Since $y_\alpha\longrightarrow y_0$, by lower semi-continuity of Φ, there exist a net $y'_\alpha\in\Phi(y_\alpha)$ such that $y'_\alpha\longrightarrow y$. This implies $\langle y_\alpha, y'_\alpha-z_\alpha\rangle\geq 0,\forall\alpha$. As $\alpha\longrightarrow\infty$, we get $\langle y_0, y-z\rangle\geq 0$. Since y is arbitrary,

$$\langle y_0, y-z\rangle\geq 0, \text{ for all } y\in\Phi(y_0),$$

which implies that $y_0\in S_1(z)$. Hence $y_0\in S_1(z)\cap B$, that is, S_1 is upper semi-continuous.

Thus the set-valued mapping S is Kakutani factorizable. Now, by Theorem 2.4, $S: K\to 2^K$ has a fixed point. That is, there exists an $x_0\in K$ such that

$$x_0\in S_1(z_0), \text{ for some } z_0\in S_0(x_0)$$

which implies that there exists $x_0\in K$ and $z_0\in\Phi(x_0)$ such that

$$p(z_0-f(x_0)) = d_p(f(x_0),\Phi(x_0)) \text{ and } \langle x_0, y-z_0\rangle\geq 0, \text{ for all } y\in\Phi(x_0).$$

It is worth noting that if $f(x_0)=z_0$, then x_0 becomes a solution to inverse quasi-variational inequality problem (1).

The following example illustrates our Theorem 3.1.

Example 3.2 *Let* $K=[-1,0]\subseteq\mathbb{R}$. *Let* $f(x)=e^x$ *and* $\Phi: K\to 2^K$ *be defined by* $\Phi(x)=[x,0]$, *for all* $x\in K$. *Here* $\mathbb{A}_{\mathrm{p}}=\{0\}$ *and it is easy to verify that all the conditions of Theorem 3.1 are satisfied and that* $x_0=0$ *is an approximate solution to inverse quasi-variational inequality problem. It is important to note that there is no solution to the inverse quasi-variational inequality problem involving these* K, f *and* Φ.

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