# A NOTE ON CERTAIN FUZZY METRIC SPACES

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Abstract. In this note we provide complete metrics for a large class of fuzzy sets on $\mathbb{R}$ which may not have bounded support and which may not even be measurable.

**1 Introduction** This note, a sequel to [9], continues to explore the problem of enlarging the scope of fuzzy numbers. Kaleva [6] consolidated the approach of Goetschel and Voxmax [5] in metrizing a larger class of fuzzy numbers, introduced by Dubois and Prade [3]. The monograph by Diamond and Kloeden [2] elaborates the contributions of Kaleva as well as their applications to differential equations besides other metrice on fuzzy sets. In the earlier publication [9], the author modified Kaleva's approach to metrize a class of fuzzy numbers that may not have bounded supports. The present note provides a method of metrizing all functions mapping $\mathbb{R}$, the real number system into [0,1], generalizing the work of Congxin Wu and Li [1].

**2 A General Representation Theorem for Fuzzy Sets** A general representation theorem for fuzzy subsets of an arbitrary nonvoid set is described below. Unlike in other representation theorems no assumptions involving either topology or convexity is made in the following.

**Proposition 2.1.** *Let $X$ be a nonvoid set and $u : X \to [0,1]$, a function such that $u(x) = 1$ for some $x \in X$. Let $C_\alpha = [u]^\alpha = \{x \in X : u(x) \geq \alpha\}$ for each $\alpha \in [0,1]$. Then*

*(i) for each $\alpha \in I$, $C_\alpha$ is a nonempty subset of $X$;*

*(ii) $C_\beta \subseteq C_\alpha$ for $0 \leq \alpha \leq \beta \leq 1$;*

*(iii) $C_\alpha = \bigcap_{i=1}^{\infty} C_{\alpha_i}$ for each sequence $\{\alpha_i\} \uparrow \alpha$ in $I$.*

*Conversely if for a nonempty set $X$, there is a family of nonempty sets $C_\alpha$, $\alpha \in [0,1]$ satisfying the properties (i), (ii) and (iii) above, then there is a unique function $u : X \to [0,1]$, viz. a fuzzy subset $u$ of $X$ such that $[u]^\alpha = C_\alpha$ for each $\alpha \in [0,1]$ with $u(x) = 1$ for some $x \in X$.*

*Proof.* Since $C_1 \neq \phi$, (i) and (ii) are clear. Let $\alpha \in [0,1]$ and $\alpha_i \uparrow \alpha$. Then $C_{\alpha_i} \supseteq C_\alpha$ for each $i$. So $\bigcap_{i=1}^{\infty} C_{\alpha_i} \supseteq C_\alpha$. If $x \in C_{\alpha_i}$, then $u(x) \geq \alpha_i$ for each $i$. So $u(x) \geq \lim \alpha_i = \alpha$. Thus $\bigcap_{i=1}^{\infty} C_{\alpha_i} \subseteq [u]^\alpha = C_\alpha$. Thus $\bigcap_{i=1}^{\infty} C_{\alpha_i} = C_\alpha$. If $\alpha = 0$, and $\alpha_i (\in [0,1]) \uparrow \alpha$, then $\alpha_i = 0$. In this case also (iii) is true.

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To prove the converse, define $u : X \to I$ by $u(x) = \sup\{\alpha \in I : x \in C_\alpha\}$. Since $C_0 = X$, $u(x)$ is well-defined for each $x \in X$. $u(x) = 1$ for some $x$ as $C_1$ is nonempty. If $x \in [u]^\alpha$, then $u(x) \geq \alpha$. Define $I_x = \{\beta \in I : x \in C_\beta\}$. Let $\alpha' = \sup I_x$. So $\alpha' = u(x) \geq \alpha$. By assumption (ii) $C_{\alpha'} \subseteq C_\alpha$. Thus $[u]^\alpha \subseteq C_\alpha$. On the other hand for $x \in C_\alpha$, $u(x) = \sup I_\alpha = \alpha' \geq \alpha$ and so $x \in [u]^\alpha$. Thus $C_\alpha \subseteq [u]^\alpha$ and so $[u]^\alpha = C_\alpha$ for each $\alpha \in I$. If for some $v : X \to I$, $[v]^\alpha = C_\alpha$ for each $\alpha \in I$, then $v(x) = u(x)$. Without loss of generality let $v(x) = r > u(x)$. Then $[v]^r = C_r \neq [u]^r$, a contradiction. So $u : X \to I$ is uniquely defined. □

We have a more general representation theorem.

**Theorem 2.2.** *Let* $X = \bigcup_{n=1}^{\infty} X_n$ *be a set where* $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n \subseteq X_{n+1} \subseteq \cdots$ *and each* $X_i$ *is nonempty. Let* $u : X \to [0,1]$ *be a function such that* $u^{[1]} = \{x \in X : u(x) \geq 1\} \cap X_1 \neq \phi$. *Then*

*(i) for each* $n$, $C_{\alpha,n} = [u]^\alpha \cap X_n \neq \phi$ *for all* $\alpha \in [0,1]$*: and* $[u]^\alpha = \bigcup_{n=1}^{\infty} C_{\alpha,n}$*;*

*(ii)* $C_{\beta,n} \subseteq C_{\alpha,n}$ *for all* $0 \leq \alpha \leq \beta \leq 1$ *for all* $n$*;*

*(iii) for* $\alpha_i \in [0,1]$ *and* $\{\alpha_i\} \uparrow \alpha \in [0,1]$, $C_{\alpha,n} = \bigcap_{i=1}^{\infty} C_{\alpha_i,n}$ *for each* $n \in \mathbb{N}$.

*Conversely if* $X$ *is the countable union of an increasing sequence of nonempty sets* $(X_n)$ *and* $\{C_{\alpha,n} : \alpha \in [0,1], n \in \mathbb{N}\}$ *is a family of nonempty subsets satisfying (i), (ii) and (iii) above then there exists a unique* $u : X \to [0,1]$ *such that for each* $\alpha \in I$ *and* $n \in \mathbb{N}$. $[u]^\alpha \cap X_n = C_{\alpha,n}$.

*Proof.* The proof of (i), (ii) and (iii) is easy and omitted.

For the proof of the converse define $u : X \to [0,1]$ by $u(x) = \sup\{\alpha \in [0,1] : x \in C_{\alpha,n}$ for the smallest $n \in \mathbb{N}\}$. Since $x \in X = \bigcup_{n=1}^{\infty} X_n$, $x \in X_{n_0}$ for the least $n_0 \in \mathbb{N}$. Let $u(x) = \alpha_0$. Then $x \in C_{\alpha_0,n_0}$. Further $[u]^\alpha = \bigcup_{n\in\mathbb{N}} C_{\alpha,n}$. Since $C_{1,1}$ is nonempty $[u]^1 \neq \phi$ and for $0 \leq \alpha \leq \beta \leq 1$. $[u]^\beta \subseteq [u]^\alpha$ as $C_{\beta,n} \subseteq C_{\alpha,n}$ for all $n \in \mathbb{N}$. If $\alpha_i \uparrow \alpha$ $\alpha_i, \alpha \in [0,1]$, then by (iii) $C_{\alpha,n} = \bigcap_{i=1}^{\infty} C_{\alpha_i,n}$ for each $n \in \mathbb{N}$. So $[u]^\alpha = \bigcup_{n\in\mathbb{N}} C_{\alpha,n} = \bigcap_{i=1}^{\infty} \bigcup_{n\in\mathbb{N}} C_{\alpha_i,n} = \bigcap_{i=1}^{\infty} [u]^{\alpha_i}$.

If $u \neq v$, then for some $x_0$ $u(x_0) > v(x_0)$ without loss of generality. So for some $r$, $u(x_0) > r > v(x_0)$. Or $[u]^r = \bigcup_{n\in N} C_{r,n} \neq [v]^r$ as $x_0 \in [u]^r$, though $x_0 \notin [v]^r$. Thus $u$ is uniquely defined. □

**3 Outer Measure Spaces** Let $X$ be a nonvoid set with a hereditary $\sigma$-algebra $\mathscr{S}$. Let $\mu^* : \mathscr{S} \to [0,\infty]$ be a nonnegative extended valued countably subadditive (set) function such that $\mu(\phi) = 0$. Such a (set) function is called an outer measure and the triple $(X, \mathscr{S}, \mu^*)$ is known as an outer measure space. An outer measure $\mu^*$ is called $\sigma$-finite if $X = \bigcup_{n=1}^{\infty} X_n$ where $X_n \in \mathscr{S}$, $\mu^*(X_n)$ is finite for each $n \in \mathbb{N}$.

It is known that an outer measure for which $\mu^*(X) < +\infty$ induces metrics naturally. The following theorem is esentially due to Frechet [5] and rediscovered by Meyer and Sprinkle

[8] (see MR 0104211 21 # 2968 for [8] by F.B. Jones). Since it is not widely known, both the statement and the proof are presented here for the sake of completeness.

**Theorem 3.1.** *Let $(X, \mathscr{S}, \mu^*)$ be an outer measure space for which $\mu^*(X) < +\infty$, $\mathscr{S}$ being a hereditary $\sigma$-algebra on a nonvoid set $X$. The functions $\rho$ and $\delta$ defined on $\mathscr{S}$ by*

$$\rho(A, B) = \mu^*(A - B) + \mu^*(B - A)$$
$$\delta(A, B) = \mu^*[(A - B) \cup (B - A)]$$

*for $A, B \in \mathscr{S}$ define pseudometrics on $\mathscr{S}$. Further $\rho$ and $\delta$ are complete pseudometrics. By defining equivalence relations $A \sim B$ in $\mathscr{S}$ if $\rho(A, B) = 0$ or $\delta(A, B) = 0$ the set of all equivalence classes in $\mathscr{S}$ becomes a complete metric space under $\rho$ or $\delta$. Also $\rho(A, B) = 0$ or $\delta(A, B) = 0$ if and only if $A \cup Z_1 = B \cup Z_2$ where $\mu^*(Z_i) = 0$ for $i = 1, 2$.*

*Proof.* Clearly $\rho(A, A)$ and $\delta(A, A) = 0$ for all $A \in \mathscr{S}$. Further $\rho(A, B) = \rho(B, A)$ and $\delta(A, B) = \delta(B, A)$ for all $A, B \in \mathscr{S}$, $\rho(A, B) = 0$ implies $\mu^*(A - B) = \mu^*(B - A) = 0$. So $A \cup B = A \cup B - A = A \cup Z_1$ with $\mu^*(Z_1) = \mu^*(B - A) = 0$ and $A \cup B = B \cup A - B = B \cup Z_2$ with $Z_2 = A - B$ and $\mu^*(Z_2) = 0$. This is true for $\delta$ as well, for $\delta(A, B) = 0$ implies $\mu^*(A - B \cup B - A) = 0$ leading to $A \cup B = A \cup Z_1 = A \cup B - A = B \cup Z_2 = B \cup A - B$ with $\mu^*(Z_i) = 0$, $i = 1, 2$ as before. If $A \cup Z_1 = B \cup Z_2$ where $m^*(Z_i) = 0$ for $i = 1, 2$, $m^*(A - B) \leq m^*(B \cup Z_2 - B) = m^*(B \cap B^c \cup B^c \cap Z_2) \leq m^*(Z_2) = 0$. Similarly $m^*(B - A) \leq m^*(A \cup Z_1 - A) \leq m^*(A \cap A^c \cup Z_1 \cap A^c) \leq m^*(Z_1) = 0$. So $\rho(A, B) = m^*(A - B) + m^*(B - A) = 0$. Similarly $\delta(A, B) = m^*(A - B \cup B - A) \leq m^*(A - B) + m^*(B - A) = 0$.

For $A, B, C \in \mathscr{S}$

$$\begin{aligned}
\rho(A, B) &= m^*(A - B) + m^*(B - A) \\
&\leq m^*(A - C \cup C - B) + m^*(B - C \cup C - A) \\
&\leq m^*(A - C) + m^*(C - B) + m^*(B - C) + m^*(C - A) \\
&= \rho(A, C) + \rho(C, B)
\end{aligned}$$

Similarly

$$\begin{aligned}
\delta(A, B) &= m^*(A - B \cup B - A) \\
&\leq m^*(A - C \cup C - B \cup B - C \cup C - A) \\
&\leq m^*(A - C \cup C - A) + m^*(C - B \cup B - C) \\
&= \delta(A, C) + \delta(C, B)
\end{aligned}$$

Thus both $\rho$ and $\delta$ are pseudometrics on $\mathscr{S}$.

We now prove that $(\mathscr{S}, \rho)$ as well as $(\mathscr{S}, \delta)$ are both complete. Let $(C_n)$ be a sequence of sets in $\mathscr{S}$. Suppose it is Cauchy in $(\mathscr{S}, \rho)$. It suffices to show that a subsequence of $(C_n)$ converges to an element $C$ in $\mathscr{S}$. Choose a subsequence $C_{n_i}$ of $C_n$ such that $\rho(C_{n_i}, C_{n_j}) < \frac{1}{2^i}$ for $i \in \mathbb{N}$ and $j > i$.

Define $D_k = \bigcap_{i=k}^{\infty} C_{n_i}$, $E_k = \bigcup_{i=k}^{\infty} C_{n_i}$ for $k \in \mathbb{N}$, $D = \bigcup_{k=1}^{\infty} D_k = \bigcup_{k=1}^{\infty} \bigcap_{i=k}^{\infty} C_{n_i} = \underline{\lim} C_{n_i}$ and $E = \bigcup_{k=1}^{\infty} E_k = \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} C_{n_i} = \overline{\lim} C_{n_i}$. As $\mathscr{S}$ is a $\sigma$-algebra, $D, E, D_k$ and $E_k \in \mathscr{S}$. For each $k \in \mathbb{N}$, $D_k \subseteq D \subseteq E \subseteq E_k$.

Now

$$m^*(E_k - C_{n_k}) = m^*\left(\bigcup_{i=1}^{\infty} C_{n_i} - C_{n_k}\right)$$
$$\leq \sum_{i=k}^{\infty} m^*(C_{n_i} - C_{n_k})$$
$$\leq \frac{1}{2^{k-1}}.$$

So $\rho(E_k, C_{n_k}) = m^*(E_k - C_{n_k}) \to 0$ as $k \to \infty$.

$$m^*(C_{n_k} - D_k) = m^*\left(C_{n_k} \cap \left(\bigcup_{i=k}^{\infty} C_{n_i}^c\right)\right)$$
$$\leq \sum_{i=k}^{\infty} m^*(C_{n_k} - C_{n_i}) \leq \sum_{i=k}^{\infty} \rho(C_{n_k}, C_{n_i})$$
$$\leq \frac{1}{2^{k-1}}$$

So $\rho(C_{n_k}, D_k) = m^*(C_{n_k} - D_k) \to 0$, as $k \to \infty$.

Since $\rho(E_k, D_k) \leq \rho(E_k, C_{n_k}) + \rho(C_{n_k}, D_k)$

$\lim\limits_{k\to\infty} \rho(E_k, D_k) = 0$

Since $\rho(E, D) \leq \rho(E_k, D_k)$ as

$D_k \subseteq D \subseteq E \subseteq E_k$ for all $k$, $\rho(E, D) = 0$

Thus $\overline{\lim} C_{n_i} = \underline{\lim} C_{n_i} = E$ or $D$ and $\rho(E, C_{n_k}) \to 0$ as $k \to \infty$.

So $(C_{n_k})$ converges to $E(= D)$ in $(\mathscr{S}, \rho)$ and hence $(C_n)$ converges to $E(= D)$ in $(\mathscr{S}, \rho)$ and hence $(C_n)$ converges to $E(= D)$ in $(\mathscr{S}, \rho)$. Thus $(\mathscr{S}, \rho)$ is complete.

If $(C_n)$ is Cauchy in $(\mathscr{S}, \delta)$ as before choose a subseqeunce $(C_{n_i})$ of $(C_n)$ such that $\delta(C_{n_i}, C_k) < \frac{1}{2^c}$ for all $k \geq n_i$.

$$\delta(E_k - C_{n_k}) = m^*\left(\bigcup_{i=k}^{\infty} C_{n_i} - C_{n_k}\right)$$
$$\leq \sum_{i=k}^{\infty} m^*(C_{n_i} - C_{n_k})$$
$$\leq \frac{1}{2^{k-1}}.$$

So $\lim\limits_{k\to\infty} \delta(E_k, C_{n_k}) = 0$

Also

$$\delta(C_{n_k}, D_k) = m^*\left(C_{n_k} \cap \left(\bigcup_{i=k}^{\infty} C_{n_i}^c\right)\right)$$
$$\leq \sum_{i=k}^{\infty} m^*(C_{n_k} - C_{n_i})$$
$$\leq \sum_{i=k}^{\infty} \delta(C_{n_k}, C_{n_i})$$
$$\leq \frac{1}{2^{k-1}}$$

Thus $\lim_{k\to\infty} \delta(C_{n_k}, D_k) = 0$. Now as $\delta(D_k, E_k) \leq \delta(D_k, C_{n_k}) + \delta(C_{n_k}, E_k)$, $\lim_{k\to\infty} \delta(D_k, E_k) = 0$. Since $\delta(E, D) = m^*(E - D) \leq m^*(E_k - D_k) = \delta(E_k, D_k)$ for all $k$, $\delta(E, D) = 0$ or $E - D$. Thus $C_{n_k}$ converges to $E = \overline{\lim} C_{n_i} = \underline{\lim} C_{n_i} = D$. So $\delta(E(= D), C_n) \to 0$ as $n \to \infty$. Thus $(\mathscr{S}, \delta)$ is complete. □

Considering the set of all equivalence classes of sets in $\mathscr{S}$ induced by the equivalence relation $A \sim B$ if and only if $\rho(A, B)' = 0$ or $\delta(A, B) = 0$ the metric induced by $\rho$ or $\delta$ is complete.

In this context we recall the following definition (see Dugundji [4]).

**Definition 3.2.** *Let $D = \{D_\lambda : \lambda \in \Lambda\}$ be a family of pseudometrics on a nonvoid set $X$. The topology $\tau(D)$ with the subbase $\{B(x; d_\lambda, \epsilon)(= \{y \in X : d_\lambda(x, y) < \epsilon\}\}$ where $\epsilon > 0$ and $d_\lambda$, a pseudometric on $X$ is called a gauge space, the family $D$ being a gauge. The gauge is called separating if for $x, y \in X$ $x \neq y$, there exists $\lambda_0 \in \Lambda$ such that $d_{\lambda_0}(x, y) > 0$. (Clearly a gauge is separating if and only if the topology is Hausdorff).*

**Definition 3.3.** *Let $(X, D)$ be a gauge space. A sequence $(x_n)$ is called Cauchy if $d_\lambda(x_m, x_n) \to 0$ as $m, n \to \infty$ for each $\lambda \in \Lambda$. The gauge space is said to be sequentially complete if every Cauchy sequence in $X$ is convergent.*

**Remark 3.4.** *A topological space is a gauge space if and only if it is a Tychonoff space. A necessary and sufficient condition for a gauge space to be metrizable is that it has a countable gauge. For these and related results Dugundji [4] may be referred.*

We have the following theorems whose straight-forward proofs are left as exercises.

**Theorem 3.5.** *Let $(X, \mathscr{S}, \mu^*)$ be an outer measure space with a hereditary $\sigma$-algebra $\mathscr{S}$. Suppose $X = \bigcup_{n=1}^{\infty} X_n$ where $X_1 \subseteq X_2 \cdots \subseteq X_n \subseteq X_{n+1} \subseteq \cdots$ with $\mu^*(X_n) < +\infty$ for all $n$. Then $(\mathscr{S}, D)((\mathscr{S}, D'))$ is a Hausdorff complete metrizable gauge space. Here $D = \{\rho_n : n \in \mathbb{N}\}$, $D' = \{\delta_n : n \in \mathbb{N}\}$ where $\rho_n(A, B) = \rho(A \cap X_n, B \cap X_n)$ and $\delta_n(A, B) = \delta(A \cap X_n, B \cap X_n)$ for $n \in \mathbb{N}$, $\rho$ and $\delta$ being the metrics defined in Theorem 3.1. Also $A, B \in \mathscr{S}$ for which $\rho_n(A, B) = 0$ ($\delta_n(A, B) = 0$) for all $n$ are identified as equal.*

**Theorem 3.6.** *Let $(X, \mathscr{S}, \mu)$ be a complete measure space. Suppose $X = \bigcup_{n=1}^{\infty} X_n$ where $X_1 \subseteq X_2 \cdots \subseteq X_n \subseteq X_{n+1} \subseteq \cdots$ with $\mu(X_n) < +\infty$ for all $n \in \mathbb{N}$. Then $(\mathscr{S}, D)((\mathscr{S}, D'))$ is a Hausdorff complete metrizable gauge space. Here $D = \{\rho_n : n \in \mathbb{N}\}$ and $D' = \{\delta_n : n \in \mathbb{N}\}$ where $\rho_n(A, B) = \mu(X_n \cap (A - B)) + \mu(X_n \cap (B - A))$ and $\delta_n(A, B) = \mu\{X_n \cap ((A - B) \cup (B - A))\}$ for each $n \in \mathbb{N}$. In $\mathscr{S}$ sets $A, B$ with $\rho_n(A, B) = 0$ ($\delta_n(A, B) = 0$) for all $n \in \mathbb{N}$ are treated equivalent.*

**4 Fuzzy Subsets of an Outer Measure Space** In this section we provide a metrical structure for a class of fuzzy subsets of an outer measure space $(X, \mathscr{S}, \mu^*)$ defined on a hereditary $\sigma$-algebra $\mathscr{S}$. This is described in Theorem 4.1 and Theorem 4.2 and Corollary 4.3 point out how a wide class of fuzzy subsets of $\mathbb{R}^n$ or $\mathbb{R}$ can be endowed with a complete metric.

**Theorem 4.1.** *Let $(X, \mathscr{S}, \mu^*)$ be an outer measure space, $\mathscr{S}$ being a hereditary $\sigma$-algebra with $X = \bigcup_{n=1}^{\infty} X_n$, where $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n \subseteq X_{n+1} \subseteq \cdots$ and $\mu^*(X_n) < +\infty$ for all $n \in \mathbb{N}$. Let $F_A^1(x)$ be the set of all functions $u : X \to [0, 1]$ such that $\{x \in X : u(x) \geq \alpha\} \in \mathscr{S}$ for* *each* $\alpha \in [0,1]$ *and* $u$ *and* $v$ *for which* $\mu^*\{x : u(x) \neq v(x)\} = 0$ *are treated as equal in* $F_A^1(x)$. *Further suppose there exists* $A \in \mathscr{S}$ *with* $A \subseteq \{x : u(x) \geq 1\} \subseteq X_1$ *for all* $u$ *in* $F_A^1(X)$ *and* $\mu^*(A) > 0$. *Define* $\Delta_n, \Delta'_n : F_A^1(X) \times F_A^1(X) \to \mathbb{R}^+$ *by* $\Delta_n(u,v) = \sup_{0\leq\alpha\leq 1} \rho([u]^\alpha, [v]^\alpha)$ *and* $\Delta'_n(u,v) = \sup_{0\leq\alpha\leq 1} \delta([u]^\alpha, [v]^\alpha)$. *Then* $\{F_A^1(X), \Delta_n : n \in N\}$ *and* $\{F_A^1(X), \Delta'_n : n \in \mathbb{N}\}$ *are Hausdorff, gauge spaces which are complete metrizable spaces.*

*Proof.* Since $\Delta_n(u,v) = \Delta_n(v,u)$ and $\Delta_n(u,v) \leq \Delta_n(u,w) + \Delta_n(w,v)$ for all $n$ each $\Delta_n$ is a pseudometric on $F_A^1(X)$. Further for $\Delta_n(u,v) = 0$ for all $n$ $[u]^\alpha = [v]^\alpha$ on $X$ for all $\alpha \in [0,1]$. Since $[u]^r = [v]^r$ for all rationals in $[0,1]$ it follows that $u = v$ almost everywhere on $X$ with respect to $\mu^*$. Thus $\{F'(U), \Delta_n : n \in \mathbb{N}\}$ is a Hausdorff gauge space. Let $u_n$ be a Cauchy sequence in $\{F_A^1(U), \Delta_n\}$. Since $\sup_{\alpha\in I} \rho(X_n \cap [u_p]^\alpha, X_n \cap [u_q]^\alpha)$ is Cauchy in $(X, \rho)$, there exists $C_n^\alpha$ such that $\sup_{\alpha \in I} \rho(X_n \cap [u_m]^\alpha, C_n^\alpha) \to 0$ as $m \to \infty$. Define $C^\alpha = \bigcup_{n=1}^{\infty} C_n^\alpha$. Clearly $C^\alpha \in \mathscr{S}$ for each $\alpha \in [0,1]$. Since $[u_m]^\beta$ for $\beta \geq \alpha$, $\alpha, \beta \in [0,1]$, for each $n \in \mathbb{N}$ $X_n \cap [u_m]^\beta \subseteq X_n \cap [u_m]^\alpha$. As $m \to \infty$, since $[u_m]^\beta \to C^\beta = \lim [u_m]^\beta$. $X_n \cap C^\beta \subseteq X_n \cap \lim[u_m]^\alpha = X_n \cap C^\alpha$. If $\alpha_i \uparrow \alpha$ in $[0,1]$, $\lim_{i\to\infty} [u_m]^{\alpha_i} \cap X_n = [u_m]^\alpha \cap X_n$. $\rho([u_m]^{\alpha_i} \cap X_n, C_n^{\alpha_i}) < \epsilon$ for all $m \geq m_0$ for all $\alpha$. Now as $\alpha_i \uparrow \alpha$, $P([u_m]^{\alpha_i} \cap X_n, [u_m]^\alpha \cap X_n) < \epsilon$ for $i \geq i_0$. So $\rho([u_m]^{\alpha_i} \cap X_n, C_n^\alpha) \leq \rho([u_m]^{\alpha_i} \cap X_n, [u_m]^\alpha \cap X_n) + \rho([u_m]^\alpha \cap X_n, C_n^\alpha) < 2\epsilon$. So as $m \to \infty$ $\rho(C_n^{\alpha_i}, C_n^\alpha) \leq 2\epsilon$ for $i \geq i_0$

Hence $C_n^{\alpha_i} \to C_n^\alpha$ as $\lim C_n^{\alpha_i} = \bigcap_{i=1}^{\infty} C_n^{\alpha_i}$. Also $[u_n]^1 \supseteq A$ for all $n$. So $\lim_{n\to\infty} [u_n]^1 = C^1 \supseteq A$. Thus $\{F^1(X), \Delta_n : n \in \mathbb{N}\}$ is a Hausdorff countable gauge space which is sequentially complete. So it is metrizable and complete. One can generate the gauge topology using the metric $\Delta(u,v) = \sum_{n=1}^{\infty} \frac{\min(1, \Delta_n(u,v))}{2^n}$ or $\sum_{n=1}^{\infty} \frac{\Delta_n(u,v)}{2^n \mu^*(X_n)}$

A similar argument shows that $F_A^1(X, \mathscr{S}, \mu^*)$ with the countable gauge $\{\Delta'_n : n \geq \mathbb{N}\}$ is Hausdorff and completely metrizable and

$$\Delta'(u,v) = \sum_{n=1}^{\infty} \frac{\min(1, \Delta'_n(u,v))}{2^n} \text{ or}$$
$$= \sum_{n=1}^{\infty} \frac{\Delta'_n(u,v)}{2^n \mu^*(X_n)}$$

gives a complete metric. □

The following theorem can be proved along similar lines.

**Theorem 4.2.** *Let* $(X, \mathscr{S}, \mu)$ *be a complete measure space where* $X = \bigcup_{n=1}^{\infty} X_n$ *where* $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n \subseteq X_{n+1} \subseteq \cdots$ *and* $0 < \mu(X_n) < +\infty$ *for all* $n$. *Let* $F_A^1(X, \mathscr{S}, \mu)$ *be the set of measurable functions mapping* $X$ *into* $[0,1]$ *such that for some* $A \subseteq X_1$ *with* $\mu(A) > 0$ *and* $u^{[1]} \supseteq A$. *Then* $F_M^1(X)$ *is a Hausdorff complete metrizable gauge space with the gauge* $\{\Delta_n : n \in N\}$ *or* $\{\Delta'_n : n \in N\}$ *where*

$$\Delta_n(u,v) = \sup_{0\leq\alpha\leq 1} \rho_n([u]^\alpha, [v]^\alpha) \text{ and}$$
$$\Delta'_n(u,v) = \sup_{0\leq\alpha\leq 1} \delta_n([u]^\alpha, [v]^\alpha)$$

*for* $u, v \in F^1_m(X)$*.* $\rho_n$ *and* $\delta_n$ *are as in Theorem 3.6.*

**Corollary 4.3.** *Let* $\mu^*$ *be the Lebesgue outer measure on* $X = \mathbb{R}^n$*. Then for any* $A \subseteq B(0;1)$ *the unit open ball with* $\mu^*(A) > 0$*,* $F^1_A(X, 2^X, \mu^*)$*, the set of all fuzzy subsets* $u$ *of* $\mathbb{R}^n$ *with* $u^{[1]} \supseteq A$ *is a sequentially complete Hausdorff gauge space with the gauge* $\{\Delta_n : n \in \mathbb{N}\}$ *or* $\{\Delta'_n : n \in \mathbb{N}\}$ *where* $\Delta_n$ *and* $\Delta'_n$ *are as in Theorem 4.1.*

**Corollary 4.4.** *If* $\mu^*$ *is the Lebesgue measure on* $\mathbb{R}^n$ *and* $A \subseteq B(0,1)$ *the unit open ball in* $\mathbb{R}^n$ *with* $\mu(A) > 0$*. Then* $F^1_\mu(\mathbb{R}^n, \mathscr{S}, \mu)$ *the set of all fuzzy Lebesgue measurable subsets of* $\mathbb{R}^n$ *with* $[u]^1 \supseteq A$ *is a sequentially complete Hausdorff gauge space with the gauge* $\{\Delta_n : n \in \mathbb{N}\}$ *or* $\{\Delta'_n : n \in \mathbb{N}\}$ *where* $\Delta_n$ *and* $\Delta'_n$ *are as in Theorem 4.2.*

**Remark 4.5.** *As the space of Lebesgue outer measurable subsets of* $(0,1)$ *or the unit ball in* $\mathbb{R}^n$ *with the metrics* $\rho$ *or* $\delta$ *induced by the Lebesgue outer measure is not separable,* $F^1_A(\mathbb{R}, 2^{\mathbb{R}})$ *or* $F^1_A(\mathbb{R}^n, 2^{\mathbb{R}^n})$ *with the gauge* $\{\Delta_n : n \in \mathbb{N}\}$ *or* $\{\Delta'_n : n \in \mathbb{N}\}$ *described in Theorem 4.1 is not separable.*

**Remark 4.6.** *Characteristic functions of singletons in* $\mathbb{R}$ *are identified with fuzzy real numbers in the Kaleva approach to fuzzy real numbers. However as singletons have zero Lebesgue measure, the characteristic functions of singletons are all equivalent to the zero function and so cannot be used to represent fuzzy real numbers in* $F^1(R)$*. However this can be remedied by considering the product metric space* $F^1(\mathbb{R}) \times \mathbb{R}$ *with the corresponding metric of the product space so that the real number system can be isometrically embedded in this product space. This is similar to embedding the real numbers isometrically in the complex plane or* $\mathbb{R}^2$*.*

**Remark 4.7.** *When* $(X, \mathscr{S}, \mu^*)$ *is a finite outer measure space, then* $F^1_U(X, \mathscr{S}, \mu^*)$*, the set of all fuzzy subsets* $u : X \to [0,1]$ *with* $[u]^1 \supseteq A$ *and* $\mu^*(A) > 0$ *is a complete metric space under the metrics*

$$\rho(u,v) = \sup_{0 \le \alpha \le 1} \{m^*([u]^\alpha - [v]^\alpha) + m^*([v]^\alpha - [u]^\alpha)]$$

$$\delta(u,v) = \sup_{0 \le \alpha \le 1} \{m^*[([u]^\alpha - [v]^\alpha) \cup ([v]^\alpha - [u]^\alpha)]$$

Finally we provide just one example to show that certain fuzzy functional equations can be solved in this setting, affording greater flexibility and scope for solving nonlinear equations involving fuzzy numbers which are neither upper semicontinuous nor convex.

**Example 4.8.** *Let* $X$ *be [0,1] and* $\mu^*$ *the Lebesgue outer measure on the power set* $2^X$ *of* $X$*. Let* $A$ *be a non-measurable subset of* $X$ *with positive outer measure and* $F^1_A(X, \mathscr{S}, \mu^*)$ *the set of all fuzzy subsets* $u : X \to [0,1]$ *such that* $[u]^1 \supseteq A$*. Define* $T : F^1_A(X) \to F^1_A(X)$ *by* $[Tu]^\alpha = [v]^\alpha = f\{x : u(x) \ge \alpha\} \cup A$ *where* $f(x) = \frac{e^{-x}+x}{2}$*. Since* $\mu^*([Tu_1]^\alpha - [Tu_2]^\alpha) + \mu^*([Tu_2]^\alpha - [Tu_1]^\alpha) \le \frac{1}{2}\mu^*([u_1]^\alpha - [u_2]^\alpha) + \mu^*([u_2]^\alpha - [u_1]^\alpha)$ *and* $F^1_A(X)$ *is complete and* $\Delta(Tu_1, Tu_2) \le \frac{1}{2}\Delta(u_1, u_2)$*,* $T$ *has a unique fixed point which is a solution of the functional equation* $Tu = u$ *in* $F^1_A(X)$*.*

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