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EXAMPLE OF CUBE SLICES THAT ARE NOT ZONOIDS

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To the memories of Som Naimpally and Joe Diestel

Abstract

Let Q be the unit cube in R^n centered at the Origin O and H a hyperplane through O.The intersection is called a central Cube slice and its study was initiated by Hadwiger, Henesley and Vaaler , continued by Ball and others. A zonoid is the range of a non atomic vector measure into R^n . In this paper, when $n = 4$ we give examples ofnon -zonoid cube slices. Let H: x + y +z +t =0 ; the slice has triangle faces and is not a zonoid. This contrasts with a result inR^3,where it follows from a classical Theorem due to Herz and Lindenstrauss that every central cube slice is a zonoid(zonotope). We also give nontrivial examples in which the slice is a zonoid. For ex. let H : ax + y + z+t=0 with a>1. If a≥ 3 , the slice is a zonotope. Otherwise it has faces that are trapeziums or pentagons and is not a zonoid.We also give other examples of the like nature.

1 Introduction

1.1 Slices Zonoids ,Zonotopes

Let us recall the result from [3]:– Let Q^n =Q =unit cube in R^n centered at Origin O; ie. Q= $\{\mathbf{x} = (\mathrm{x}_k) : |x_k| \leq \frac{1}{2}\}$.

Let H be a vector subspace of dimension n-1, ie. a plane thru the Origin with equation : H= (x= (x_k) with x .a =0) for a(non zero) vector a in R^n. The intersection of H andQ will be called central slice or, slice. Following [3] we denote by$|A|$ the appropriate volume /area of the measurable set A ,and assume n ≥2. As other examples let us note the papers [7], [8],[13]initial to this subject , and the surveys [5], [10] [14]]that treats many related topicsWenote the p th powerof L^p norm of the sinc function in [3] : for (p≥ 2):

$$I_p = \frac{1}{\pi} \int\limits_R \frac{|sint|^p}{|t||^p} dt \tag{1}$$

An upperbound for this is:- $\frac{\sqrt{2}}{\sqrt{p}}$, with equality iff $p = 2$. The lower bound is assumed by H: x_k= 0 and upper only if n=2 and H: $x+y$=0 or with $x-y$=0

Let us mention that Valler[13] considers concepts of analytic interest; his results not only prove lower bound but also apply to Minkowski's Theorem on Linear foirms.

We note that there are alsoresults on sections by centralplanes of dimension k(see [14] TH 1.2, 1.3 p 154), alsodue toBall. We treat only the case k= n-1.

This estimate is in [3]; see also [10, Ch1]. The proof of this estimatein [3, p468] is with " direct " and uses only elementary methods . The one in [10] uses Fourier methods.This integral I_p has found use in wavelets[11]

For our needs we use the more precise values also from ([3] Lemma3) below ,see eq(9), (10). In [3] this is derived ,first using Characteristic functions(= Fourier Transform) then the standard Inverse Fourier Formula .

As pointed out by an anonymous referee (of another paper)– see Acknowledgements –this I_4 is in the classic, [12] (also in [9]); see [10] for many related deeper results However we use the formula from [3] for vol of slice of cube . Our interest is more in the sliceitself .With n=4 in Sec 3we give example ofa a (central)slice that has a triangle face, and isnot nota zonoid ("$face$") defined below).On the other hand,in Sec 4 we give examples of slices that are zonoids, and othersthat have a pentagon or trapezium face and so are not.

Notation and preliminaries : We write an element of R^4 as (x, y, z, t) and use a, b, c, d as coefficients. Below we avoid the case when H is paralll to acoordinate hyperplane;inthis case the slice is a Cube of lower dimension and so a zonoid

Let a hyperplane be H : ax + by + cz + dt =0 .As in [8], we may assume that no coefficient is zero, and next they are all positive. Further we may assumethat $a \geq b \geq c \geq d$ and then by dividing by d, that H: ax + by + cz + t =0 with $a \geq b \geq c \geq 1$. In all examples of non zonoids we consider the equation [t= -1/2] to get a Face that is atriangle,trapezium or pentagon(disqualifying slice fombeig a zonoid : see beginning of Sec 3).

In ex 3.1 we consider the case when a=b=c(= 1); and as mentioned above show that the slice has triangular faces and so not a zonoid(" face" defined below).The sections of this sliceby planes[t= -c] with $0 < c < 1/2$ are hexagons .These tend to the triangle face as c tends to 1/2 . We may feel that " cube slices in R^4 are never nontrivial zonoids". Hencein ex 4.1 we consider H: a x + y + z + t =0 with $a > 1$ Now the slice is a zonoid if $a \geq 3$ and is a paralletope;in the contrary cases the slice has pentagon faces and isnot a zonoid . In Ex4.2 we consider H:a(x +y)+ z +t=0 ; the slice is not a zonoid onaccount of trapezium faces.In Ex 4.3 we have H: a(x + y) +z +t=0and slice has pentagon faces In ex4.4 Webriefly indicate special cases of H: ax + by + z +t=0 with$a > b >$1. As the methods in these examples is same asthe one in Ex 3.1 we donot give details. In Ex.4.4 we consider the case of H:$ax + by + cz + t = 0$.Slice is a paralleotope in case $a \geq b + c + 1$ and $b \geq c + 1$. If (i) and (ii) both fail then the slice has pentagon faces and is not a zonoid

We give these as samples ; and do not consider every possible case .Roughly , the non zonoids prevail in our list of examples.

DiagramThey will help.

Our **methods** are elementary and can be found for ex in [6]. We do use the formula for vol of slices from [3] (see also [10] ch1) referred to above.

We notethatthat in al of our examples we use the face [t= -1/2] of the cube , this is also a faceof the slice The "*domain*" C of face is found first , then an affine map T to determine theFace T(C). The points in C are found by checking the x and y intercepts of lines involved satisfy theconditions for slice:–$|x|$, $|y|$ and $|z|$ are all$\leq 1/2$. This condition must be satisfied by all coordinates of points in the Face (of slices) that we find, and leads tothe conditions imposed on the coefficients of H.

Let us first describe the result on slices

. **1.2Theorem ([3][5] ,[7], [8] [13] ,[14])**In all dimensions the measure of cube slice is between 1 and $\sqrt{2}$; these arebest.

1.3Zonoids

Returning to the title of this paper, about Zonoids:– Our concern is :– When is a slice a zonoid? We do not have a complete characterization of this .Instead let us concentrate in R^4, and give examples of non zonoid slices as well as those that are zonoids and a consequence(known) for I_4 . We recall from [6] with X=R^n :– . A zonoid is range of a non atomic vector measure and above all the classical Liapunovs Theorem:- A zonoid is compactand convex A zonotope is sum of segments(each centered at the Origin) For our purpose we need the classic result of Herz and Lindenstrauss from [6]:– The closed unit ball in every 2dimensional normed space is a zonoid

2 Zonoids and Zonotopes

2.1 Theorem[6]

i)If H is 2 dimensional then every such slice is a zonotope ii) In all dimensions every projection of Q is a zonotope

Proof:

(i) This follows from the classic result due to Herz and Lindenstrauss quoted above and the result from ([6]) :- inR^2 every centrally symmetric polygon is always a sum of segments

(ii) This is in [6] and can also be verified directly. Hence the Theorem.

Remark 2.2:

For much more about projections see, [4].

3 Example of slice that is not a zonoid

A notedbefore,in contrast(Th 2.1, part i) to the situation in R^3 we offer an example of a slice in R^4 that is not a zonoid. Reasons to disqualify it from being a zonoid are the useful facts, all from [6] :–If K is a zonoid then

(i) K has center of symmetry c say .In fact by definition of " K is a Zonoid "(see Introduction)

K=$\mu(\sum)$ for a (vector measure) μ then c= $\frac{1}{2}\mu(S)$ will do For, with $A^c =$ complement of set A, we have $\frac{1}{2}(\mu(A) + \mu(A^c) = 1/2\mu(S)$ =c for every A in domain$\sum$ of μ

(ii) faces are translates of zonoids of lower dimension and

(iii) Since it has no center of symmetry,the triangle is not a zonoid; neither is a trapezium (trapezoid) or a pentagon

(iv)Hence any compact ,convex, balanced set that has a triangular,(or a trapezium face) cannot be a zonoid. Thus, the Octohedron in R^3 is nota zonoid, for it has triangular faces. There are deeper non zonoids for ex the 1976 result due to LE Dor (for ex[10]):–If $1 < p < 2$ and $n\geq 3$ then the closed unit Balls of the spaces l^p_n are not zonoiids

We give, in Th3.4, a version of(ii) from [2]:– a face (defined below) is a translate of some zonoid of lower dimension . We need this version in the Th 3.4 to produce non zonoid slices in our examples.

Let us recall fom [6] theterm , Face of acompact convex set K in a real(normed space) X. Let us use H for any hyperplane (not necessarily thru O)

As above a hyperplane is

$$H = (x\epsilon X : (x, x^*) = \alpha) \ , \tag{2}$$

where x^* isa non zero functional in X^* and $\alpha$is a real number .

The set K is on one side" of this H if

$$sup[(x, x*) : x\epsilon K]\leq\alpha, \tag{3}$$

A similar condition holds with inf replacing sup and by$\geq$ replacing $\leq$;and H supports K if K is to one side of H as in eq(3,) H $\cap K\neq\phi$and K is not entirely contained in H. Finally the (compact convex) set H$\cap$ K is called a Face of K .

Below we use the fact that an affinemap preserves convexity.

Let X, Y be real Banach spaces .Then a mapT: X Y is affineif T (ax + by)= aT(x) + bT(y) for every x, y in X and a , b 0 with a + b =1.ie; the definition of Linear map is now restricted to line segments in domain.

Let us recall ; K= $H\cap Q$ is the slice correponding to H[$t = -1/2$] In the next (and other) examples all we need is that the relevant , y, z coordinates of our pints are limited by $|x| \leq 1/2$ etc .

3.1Example with triangle face

Let us recall H is given in R^4 by

$$x + y + z + t = 0, \tag{4}$$

The slice (i) has triangular faces and so is not a zonoid

(ii) the intersections of slice with t= -c , 0$\leq c < \frac{1}{2}$

are hexagons ;these are sections (iii) These tend to the above triangle as c tends to 1/2

proof(i)

Substituting t = -1/2 in eq(4) of H , for any **x** =(x, y, z, t) in this H we have

x = x(1,0,0 -1) + y (0,1,0,-1) +z(0,0, 1, -1) is the linear combination x u + y v + z w. (these 3 vectors u, v, w are Linearly independent)

First consider the 2 dimensional set S in slice,in span of vectors u and v. Starting with A (u/2) on the x axis and going counterclockwise,we see

that S is a hexagon with vertices A (u/2), B(v/2), C((v-u)/2, A'= -A,

B'= -B , C= -C. Further it is regular all sides have length $1/\sqrt{2}$ and that this = sum of 3 segments , OA , OC and OB'. This set S is in plane z=0 Now we consider the 3rd term in above eq for x ; we note that the vector D= w/2 cannot be added to A or B as the sum will leave the cube We consider the Hyperplane $H_1 =$ (x, y, z, t): t= -1/2) or, simply by t = -1/2 and claim that

(a) this plane supports the slice K and that

(b)the face F =$H_1 \cap K$ is convex triangle.

As noted above, (b) disqualifies the slice from being a zonoid

Let us verify the claims. Now (a) follows directly from def. of Q.In fact for every element in Q we have $t \geq -1/2$ ie., Q is to one side of H_1; so is the slice .Further , the elements A , B, are in the slice, and also lie inH_1 , hence in Face F. The Origin O is in slice K not in H_1; ie. the sliceis not entirely contained inH_1.Hence H_1 is a supporting hyperplane of the slice as claimed.

For claim (b) we may write any element in the Face as

x = (x, y, z, t)=(x, y, $\frac{1}{2} - x - y, -1/2)$,

since we use t= - 1/2 in eq (4) of H and

we get z= $1/2 - -x - y$.

As x is in Q we need $|x|$ and $|y|$ and also $|z|$ from above ≤1/2 and so

$$|1/2 - x - y| \leq 1/2 \ , \tag{5}$$

This last translates to

$$0 \leq x + y \leq 1 \ , \tag{6}$$

Geometrically, we note that the last inequality gives two boundary lines of "*domainC*" sayL_1 := x+y =1 , and L_2:= x+ y=0. We sketch these lines; as the x-intercept ofL_1 exceed the bound 1/2 let us consider its intersection with the line x =1/2 toget point (1/2, 1/2).; intersection of L_2 with the line y=1/2 gives (-1/2, 1/2). This line with y=-1/2 gives (1/2, -1/2)

These result in a (convex right angled) triangle C in x-y plane with above vertices P(1/2, -1/2) , Q(1/2, 1/2) and R (- 1/2, 1/2)

Now let us define a map Tfrom Cto F by

$$T(x, y) = (x, y, 1/2 - x - y, -1/2) \tag{7}$$

and C is its domain. Then we may verify that, T is affine and that T(C)=F. Further as observed before statement of this example, affine map preserves convexity, and so the image T (C) = convex hull of the 3 points

(p_1,p_2 , p_3) where p_1 = T(P)= (1 /2 , -1/2 ,1/2, -1/2) , p_2 =

T(Q)= (1 /2, 1/2, -1/2, -1/2) andp_3= T(R)= (- 1 /2, ,1/2, 1 /2,-1/2). These points are not collinear, form a triangle and we conclude that the face F is a triangle , completing Claim (b) and proof of (i) .

We need to prove (ii) and (iii) .

Recall K = slice ; now we let $0 < c < 1/2$ and Section K_c = K ∩[t = - c].

Use t= - c in eq (4) H; any x in Kc is then of

the form $\mathbf{x}$ = (x, y , -x-y+c, -c) with the conditions

$|x|$, $|y|$ and $|x + y - c| \leq 1/2$.

Similarly to above (6) this last translates to

$$-1/2 + c \leq x + y \leq 1/2 + c \ , \tag{8}$$

As in part (i) we draw the "*boundary*" lines L_1 , L_2 from eq (8) . Again , both the x and y- intercepts of L_1 fail the bounds of 1/2; however L_2 passes (noting the limits on c)Then we find the vertices of our "*domainC*" by intersecting L_1 and L_2 with the lines y=1/2, y= -1/2, x= 1/2 and x= - 1/2. . We get a hexagon(domain). Its 6 vertices are shown in a Chart in next Theorem 3.3 and as follows:–

p_6 =(c, -1/2) on lines y = -1/2 and L_2 , p_1 =

(1 / 2, -1/2) andp_2 =(1/2 , c) on line x= 1 / 2 andL_1 .Next p_3 =(c, 1 / 2) on lines L_1 and y=1/2 and p_4 =(- 1/2, 1/2) then p_5 = (-1/2, c) on lines L_2 and x= -1/2

These 6 points (p_i) form a hexagon making

the new domain C of map T defined analogous to eq(7) in part (i) above.

As there we see that the section K_c = T(C) is also a hexagon.

Finally let c tend to 1 / 2 ; then we see from above that the

following vertices coincide:- $p_2 = p_3$ =(1/2, 1/2) ,$p_4 = p_5$

=(-1/2, 1 /2) and p_6 = (1/2, -1/2) = p_1 . Correspondingly (as in

case i above) we verify that the section T(C) becomes the triangle in

part(i) completing thereby proof of (ii) and the example

Remark 3.2. Above we used the hyper plane given by the equation,

t= - 1 /2 and found that the face of slice given by it is triangular

; we may instead consider t = 1 / 2 Further, the equation defining H is

symmetric with respect to the four variables x, y, z, t. Hence we may conclude that there are 8 triangular faces. We do not know what are the remaining faces and we think there are 4 more but not triangles .

For the next result, we follow [3]Lemma 3 (see also[10] ch1) and recall from Introduction eq(1) theintegral I_p :

$\frac{1}{\pi}\int_R \frac{|sint|^p}{|t||^p} dt.$

Here p is an integer ≥2, and we have from the result in [3] above,

the formula forthe exact value of slice :-

$$\|H \cap Q| = \frac{1}{\pi}\int_R g(t)dt, \tag{9}$$

where g is the finite product

$$g(t) = \prod_{1}^{N} \frac{sin a_i t}{a_i t}, \tag{10}$$

and the sequence (a_i) (of coordinatesof vector normal to H) is normalized in l_2 andalso each $|a_j| \leq 1/2$

To find volume of slice S we use Cavaleri s principle = Fubini's Theorem .

Let $|A(c)|$= area of the section of S by plane [t= -c]. Then the vol of slice =2 $\int_0^{1/2} |A(c)| dc$.

We saw in ex 3.1 that A (c) is a hexagon . We give the details in the next result;

3.3Theorem (i)The volume of the slice in Ex3.1 is 4/3 (ii) $I_4 = 2/3$

Proof(i) We refer to part (b)in ex3.1 and list the vertices of the hexagons in domain C as well as in the range T(C).

Recall T(x, y)= (x, y, c-x-y, -c) with $0 < c < 1/2$.

Domain C .... rangeT(C)

p_1(1/2, -1/2)................... P_1(1/2, -1/2, c, -c)

p_2 (1/2, c) .... P_2 (1/2, c ,-1/2 , -c)

p_3 (c , 1/2) .... P_3 (c , 1/2, -1/2, -c)

p_4 (-1/2, 1/2) .... $P_4(-1/2, 1/2, c, -c)$

$p_5(-1/2, c)$..........................P$_5$(-1/2, c, 1/2, -c)

p_6(c, -1/2) ....P_6 (c, -1/2, 1/2 , -c)

We claim that area of domain C=

$$|A(c)| = (3/4 - c^2), \tag{11}$$

In the following we use formula for area of trapezium by rule (1 /2) h (a + b) where h is the height and a, b are lengths of parallel sides.

Let us use the chart for domain C first then use it to get the image.

The domain C = two trapeziumsT_1 andT_2 ; these are the top and at bottom respy. Namely, T_1 has vertices, p_5, p_2, p_3 and p_4 and T_2 has vertices, p_6, p_1,p_2 andp_5.

Then we have

$|T_1|$= $\frac{1}{2}(1 + c + 1/2)(1/2 - c) = 1/2(3/2 + c)(1/2 - c) and$

$|T_2|$= 1/2 (1 -c +1/2) (1/2 + c)= 1 /2(3/2 -c) (1/2 +c)

Adding them we get the eq (11) for A(c).

To get the area of image T(C) observe that the domain C is the projection on plane(z=0) of the wanted T(C) .

For the factor needed we note that the unit normal to H is n=(1/2 , 1/2 , 1 /2 , 1 /2) and that e_3 = (0,0,1, 0).

Using the dot product $n.e_3$ we see that area of Projection = 1/2 area of T (C). Thus the area of T (C) = $2(3/4 - c^2)$ from above.

We integrate from c=0 to 1 / 2 to get

$2\int_0^{1/2}$ ($\frac{3}{4}$ - c ^{2})$dc = 2/3$

Taking into account alsothe part t= 1/2 to 0 we get 2(2/3) =4/3 as claimed

Part (ii) :Recalling H: x+ y + z+t =0 and the coefficients, normalized , we apply the formula from [3] quotedabove in eq (9) , (10) to get vol of slice=

$\frac{1}{\pi}\int_R \frac{(sint/2)^4}{(t/2)^4}dt.$

(as in part (i) we used each a_i coefficient is 1/2 due to normalizing them in eq of H) Now a change of variable gives

$\frac{2}{\pi}\int_R (sint/t)^4$ dt = $2I_4$.

From part (i) we have $2I_4$ = 4/ 3

and so part (ii) and the Theorem

Above,in example of a non zonoid slice we used the important fact about faces of a zonoid from [6] (innext Theorem) The following proof is different from the one in[6] which uses Every Zonoid is a zonoid of moments .This approach is not suitable for our purpose; hence we give a proof (in[2]) in next result. We see in the proof that it is more meaningful incase the Face isnot a singleton, ie. when the composed measure $x^*o\mu$ is not equivalent to μ

3.4Theorem[6][2]]Let K =$\mu(\sum)$ be a zonoid in X=R^n, and H a supporting Hyperplane given by x* in X*.Then the face F=$K\cap H$ is a translate of a zonoid of lower dimension . In fact there are μ almost disjoint sets S_0 and S_1 such that

(i)$x^*o\mu(S_1)$=sup$\{x^*o\mu(E) : E\epsilon\sum\}$ and every set E inS_0 is $x^*o\mu$ - null

(ii F= $\mu(S_1)$ + $\mu_{S0}(\sum)$.

Proof: With β= sup$x^*\mu(\sum)$ we have,from definition of F

$$F = \{x : x = \mu(E) s.t. x^*(x) = \beta\}, \qquad (12)$$

Let S^+ be such that $x^*o\mu(S^+) = \beta$.

We will writeS^+ =S_0 ∪ S_1 as stated in the Theorem.

To do this let usnote that the signed measure $x^*o\mu << \mu$

;consider those E that are $x^*o\mu$ - null but not $\mu - null$.

(ifthere are no such sets E then S_0may be taken to be $\emptyset$).

Othewise consider a maximal pairwise disjoint family ofsuch sets; this family is countable, so that theirunion is in $\sum$. Call this set S_0 and let $S_1 = S^+ - S_0$ .Then

(i) follows from the fact that

x^*o $\mu(S_0)$ =0 by the construction of S_0 and so

β = x *o $\mu(S^+)$ = x * o$\mu(S_0)$+x *$o\mu(S_1)$

= x^*o $\mu(S_1)$. we see that second part in (i) follows by construction again.

As for part(ii) we have from Eq (12) if x= $\mu(E)$ ϵ F then

$x^*o\mu$ (E) = β .

We need to write x =μ (E) asthe sum, μ (E)= μ (S_1) + μ (A) for some set A⊂S_0 To do this,first we claim that(ae—x*$o\mu$) this E⊂ S^+ . Ifnot we can argue to contradict to the fact that S = S^+ ∪S^- is a Hahndecomposition of the underlying set S in terms of $x^*o\mu$

Again we can argue that S_1 - E is μ null;from it being $x^*o\mu$ null, and then on (subsets of) S_1 these two measures are equivalent by construction.

Hence we have E= E $\cap$ S_1 $\cup E\cap S_0, andso$
$\mu(E) = \mu(E\cap)S_1) + \mu(E\cap S_0) =$
$\mu(S_1) + \mu(A) with A = (E\cap S_0)\subset$$S_0$ as claimed.
Hence the Theorem

4 Examples of non zonoids with pentagon faces and some zonoids a >1

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As in Introduction welet H: ax + by+ cz + t=0 be a hyperplane in R^4,with a$\geq$b $\geq$c $\geq$1.

We donot consider all cases but hope the following are of interest. There are non trivial cases of zonoid slices . As the methods are same as the one in the earlier ex 3.1 we only summarise the results It seems the non zonoid slices dominate:–

In the next ex. we donotknow if the converse is truein this gnerality. Hencewe give some special cases ofthe eq of H in the EXs 4.2 and on.In all cases for the Face we use as before the support hyperplane of Q [t= $\frac{-1}{2}$]

4.1 H: general caseabove

If a $\geq b + c + 1$ then the sliceis a zonotope.

Proceeding as in Ex3.1, we find the " domain" for the face .For this we have the boundary lines L_1 to be ax + by = $\frac{c+1}{2}$ and L_2 to be ax+by =$\frac{-c+1}{2}$

Firstwe note bothx andy-intercepts of L_2 are always (regardless ofthis condition) $\frac{1}{2}$ in absolute value. As forL_1 this condition gives the x-intercepttobe $\leq$ $\frac{1}{2}$ in absolute velue. In the following " domain the vertex p_2 depends on this condition, ie. its " $|x|$: satisfies the limits $\leq \frac{1}{2}$.

Withthe condition above we have now the chart

Domain

p_1 $(\frac{b-c+1}{2a}$, $\frac{-1}{2}$)

p_2 $(\frac{c+1+b}{2a}$, $\frac{-1}{2}$)

p_3 $(\frac{c+1-b}{2a}, \frac{1}{2}$)

p_4 $(\frac{1-b-c}{2a}$, $\frac{1}{2}$)

Next the corresponding points on the Face:–

FaceT(x,y)=(x, y, $\frac{1/2 - ax - by}{c}$, $\frac{-1}{2}$)

P_1 $(\frac{b-c+1}{2a}, \frac{-1}{2}, \frac{1}{2}, \frac{-1}{2})$

P_2 $(\frac{c+1+b}{2a}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2})$

P_3 $(\frac{c+1-b}{2a}, \frac{1}{2}, \frac{-1}{2}, \frac{-1}{2})$
P_4 $(\frac{1-b-c}{2a}, \frac{1}{2}, \frac{1}{2}, \frac{-1}{2})$

It is seen that this domain is a parallogram with the parallel sides(so is the Face):

$(p_1p_2) = (p_4p_3) = (\frac{c}{a}, 0)$ and

$(p_2\ p_3)$ = (p$_1$ p_4) =($\frac{b}{a}$. -1)

Likewise, it can be verified using the map "T" , that so is the Face.

Further, the sections of thesliceby planes with eqs t= $-c_1$, with 0 <c$_1$ < 1/2 are parallograms that are congruent to the one for the Face. Hence it follows (using symmetry) thatthe slice is a zonotope.

4.2 H : a x +y + z+ t=0

In one direction this is a special caseofEx 4.1 ;however due tolimitation of eq ofH we can state " iff" and we give details :–

In this case if $a \geq 3$ then the slice isa zonoid ; it is a paralleotope if not the slice has pentagon faces and is not a zonoid.

Case $a \geq 3$: Face is a ;paralleogram ; so is every parallel section congruent to it

Letus note that analogously to eq(6) above we replace x by ax there. Thus the x-intercept of the line with equation $ax + y = 1$ is x= $1/a$.The condition $x \leq$ 1/2 now holds (due tothe condition on a). This forces the "Domain" to be a paralleogram as we now state. As before we use (x,y) forpoints p_i and

x= T(x,y)= (x, y, 1/2- (ax +y), -1/2) for points P_i :

Domain C(x,y) Face T(C)

p_1(1/2a, -1/2) P_1(1/2a, - 1/2, 1/2, -1/2)
p_2(3/2a, -1/2) P_2 (3/2a, -1/2, -1/2, -1/2)
p_3 (1/2a, 1/2) P_3(1/2a, 1/2, - 1/2, -1/2)
p_4 (-1/2a, 1/2) P_4(- 1/2a, 1/2, 1/2, -1/2)

We see that the opposite sides are parallel and have equal length , so that the Face is a rhombus .Further so is any section by plane[t= -c] with $0 < c < 1/2$, the area does not depend on c and equals $\sqrt{1 + 2a^{-2}}$

case $a < 3$ in this case we can verify the " domain" tobe apentagon; so is the face and slice is not a zoniod

4.3 H: a(x +y) +z +t =0 with $a \geq 2$

The Face$[t = -1/2]$ is a trapezium again, slice not a zonoid

4.4 H:ax +by+ z +t =0 (compare ex4.1) Face is a paralleogram in case $b + 2 \leq a$. The parallel sections$[t = -c]$ are congruent parallograms, and the slice isa parallotope. Otherwise Face is a pentagon, slice is not a zonoid

4.5 H: ax + by + z +t =0 The slice is a zonotope if (i) a $\geq b + c + 1$ and (ii) $b \geq c + 1$.

In case (i) and (ii) both fail Face is a pentagon and slice is nota zonoid.

If (i) fails but(ii) is true then the Face is a hexagon

Remark 4.5 In the last case we dont know if the slice is a zonoid

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