A NEW CLASS IN BCK-ALGEBRAS

Dedicated to the late professor Shôtarô Tanaka (1928-2019)

Masaaki Kumazawa

ABSTRACT. In BCK-algebras, it is well known that a commutative BCK-algebra is as a lower semilattice with respect to the operation $\wedge$. In this paper, we show that a new class in BCK-algebras which is a proper large class than the class of the commutative BCK-algebras exists, and this class is as a lower semilattice with respect to the new operation $\times$.

1 Introduction

A BCK-algebra is a generalization of the following two concepts. It is generalized from, one hand the concept of the algebra of sets only with the set-difference (see W. Sierpiński [5]), the other hand the concept of the propositional calculi which contain the only implication functor among the logical functors (Meredith's System B-C-K, see A. N. Prior [4]).

BCK-algebras were introduced by K. Iséki in the article [2].

These algebras are partially ordered sets. Further, S. Tanaka showed that the commutative BCK-algebras of a special class in BCK-algebras are lower semilattices with respect to the operation $\wedge$ in the article [6].

In this paper, we will define a new concept which is called the condition $(\mathrm{I})_{x,y}$ for x, y in a BCK-algebra X. Using this concept, we shall clarify a difference between the existence of a commutative element $x \wedge y = y \wedge x$ and the existence of the greatest lower bound of x and y in a BCK-algebra X. And more, we will define the BCK-algebras with Condition (I) X which is a special class satisfying the single condition $(\mathrm{I})_{x,y}$ for any x, y in X. We shall show that the BCK-algebras with Condition (I) is a proper large class than the class of the commutative BCK-algebras, and this class is as a class of lower semilattices in BCK-algebras with respect to the new operation $\times$.

2 Preliminaries and Problems

We will start out to recall the definition and some basic properties of BCK-algebras.

Definition 2.1 An algebra $X =< X; *, 0 >$ of type$< 2, 0 >$ satisfying the following five conditions is called *a BCK-algebra* :

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For any x, y, z in X,

(I) $\{(x*y)*(x*z)\}*(z*y)=0$,
(II) $\{x*(x*y)\}*y=0$,
(III) $x*x=0$,
(IV) $0*x=0$,
(V) $x*y=0,\ y*x=0$ imply $x=y$.

In this algebra, we denote $x \leqq y$ when $x*y=0$.

We will state basic properties of BCK-algebras.

Proposition 2.2 *For any x, y, z in a BCK-algebra X, we have the following properties hold.*

(1) *$X=<X;\leqq>$ is a partially ordered set with respect to $\leqq$.*
(2) *$x \leqq y$ implies $x*z \leqq y*z$, $z*y \leqq z*x$.*
(3) *$(x*y)*z=(x*z)*y$.*
(4) *$(x*y)*(z*y) \leqq x*z$.*
(5) *$x*y \leqq z$ implies $x*z \leqq y$.*
(6) *$x*y \leqq x$.*
(7) *$x*0=x$.*

For details of the proofs, see K. Iséki, S. Tanaka [3].

We define the operation $x \wedge y$ by $y*(y*x)$. A BCK-algebra X is said to be *commutative* when $x \wedge y = y \wedge x$ holds for any x, y in X.

For a commutative BCK-algebra X, the following theorem holds.

Theorem 2.3(Tanaka's Theorem)(S. Tanaka [6], K. Iséki, S. Tanaka [3]) *Any commutative BCK-algebra $X=<X;*,0>$ is a lower semilattice with respect to the operation $\wedge$ in X.*

This theorem asserts that, if a BCK-algebra X is commutative, then the greatest lower bound exists for x and y in X, and is identical to $x \wedge y$.

Inspired by this theorem, we consider the following problem.

Problem Under what condition on x and y in a non-commutative BCK-algebra X, does there exist the greatest lower bound of x and y, and when is obtained by $x \wedge y$?

3 Basic properties of the Condition $(\mathrm{I})_{x,y}$ in a BCK-algebra

We will give an additional condition in BCK-algebras.

For x, y in a BCK-algebra X, we put the condition $(\mathrm{I})_{x,y}$ in the following.

Condition $(\mathrm{I})_{x,y}$ For x, y in a BCK-algebra X, z exist in X, we say that z *satisfies the condition* $(\mathrm{I})_{x,y}$ when z satisfies the following conditions (i)~(iii).

(i) $z \leqq x,\ z \leqq y$,
(ii) $x*z \leqq x*y$,

(iii) $\quad y * z \leqq y * x.$

Under the condition $(\mathrm{I})_{x,y}$, the following basic properties hold for z in a BCK-algebra X.

Proposition 3.1 *If z is the greatest lower bound of x and y in X, then z satisfies the condition* $(\mathrm{I})_{x,y}$.

Proof Suppose that z is the greatest lower bound of x and y in X. Clearly, z satisfies the inequalities (i). We will show the inequality (ii).

Now, we will show that $y \wedge x$ is a common lower bound of x and y. By (6) in Proposition 2.2, (II) in Definition 2.1, we obtain

$$y \wedge x = x * (x * y) \leqq x, \; y \wedge x = x * (x * y) \leqq y \tag{3.1}$$

Then, $y \wedge x$ is a common lower bound of x and y.

Let $0 \neq u \in X$ be a common lower bound of x and y. Here, we will show that

$$y \wedge x \leqq u \tag{3.2}$$

for any u in X.

First, if $y \wedge x > u$ implies,

$$u * (y \wedge x) = 0 \tag{3.3}$$

On the other hand, by (2), (6) in Proposition 2.2,

$$u * (y \wedge x) \leqq u * (x * x) = u \neq 0 \tag{3.4}$$

Hence, (3.3) contradicts (3.4).

Second, if $y \wedge x \ngeqq u$, $y \wedge x \nleqq u$ imply that $y \wedge x$ and u are anti-chain. Then, two maximal lower bound of x and y exist. However, z is the greatest lower bound of x and y. This is contradiction. Hence, we obtain (3.2).

Here, z is the special element in the set of the common lower bounds of x and y, by (3.2), we have

$$(x * z) * (x * y) = (y \wedge x) * z = 0. \tag{3.5}$$

Therefore, we will prove the inequality (ii). By the same way, we have the inequality (iii).

□

Proposition 3.2 *If $x \wedge y = y \wedge x$, then $x \wedge y$ satisfies the condition* $(\mathrm{I})_{x,y}$.

Proof Noting that $x \wedge y$ is a common lower bound of x and y. This implies the inequalities (i).

We will show the inequality (ii). Put

$$z = x \wedge y = y \wedge x.$$

By (3) in Proposition 2.2,

$$
\begin{aligned}
(x * z) * (x * y) &= \{x * (y \wedge x)\} * (x * y) \\
&= \{x * (x * y)\} * (y \wedge x) \\
&= (y \wedge x) * (y \wedge x) \\
&= 0.
\end{aligned}
$$

This implies the inequality (ii). By the same way, we have the inequality (iii).

□

We will give a necessary and sufficient condition on (ii) in the condition $(\mathrm{I})_{x,y}$.

Proposition 3.3 *Let $z \leqq x,\ z \leqq y$. Then the following the conditions (1)∼ (3) are equivalent.*

(1) $x * z \leqq x * y,$
(2) $x * z = x * y,$
(3) $y \wedge x \leqq z.$

Proof Clearly, the condition (2) implies (1). In a BCK-algebra, from (2) in Proposition 2.2, $z \leqq y$ implies $x * y \leqq x * z$. Hence $x * z = x * y$. Then the conditions (1) and (2) are equivalent.

The inequality (1) means

$$(x * z) * (x * y) = 0. \tag{3.6}$$

On the other hand, by (3) in Proposition 2.2, the condition (3) implies

$$(y \wedge x) * z = \{x * (x * y)\} * z = (x * z) * (x * y) = 0. \tag{3.7}$$

The equalities (3.6), (3.7) imply that the condition (1) and the condition (3) are equivalent.

□

It is more convenient to substitute the condition (3) for the condition (1). This substitution is often used in the proofs.

Let $I(x, y)$ denote the set of elements satisfying the condition $(\mathrm{I})_{x,y}$.

Proposition 3.4 *$I(x, y) \neq \phi$, then there exists the greatest lower bound of x and y in $I(x, y)$.*

Proof Assume that z exists in $I(x, y) \neq \phi$.

For any u in X such that $u \leqq x,\ u \leqq y$, we will show $u \leqq z$.
By the assumption, z satisfies the condition $(\mathrm{I})_{x,y}$. From Proposition 3.3,

$$y \wedge x \leqq z. \tag{3.8}$$

Further, by (2) in Proposition 2.2 and (3.8),

$$u * z \leqq u * (y \wedge x). \tag{3.9}$$

On the other hand, $y \wedge x$ is a common lower bound of x and y. By (3.9), replace u with $y \wedge x$,

$$u * z \leq (y \wedge x) * (y \wedge x) = 0.$$

Then, we get $u \leqq z$. Therefore, z is the greatest lower bound for x and y.

□

Next, we shall prove an important proposition which is applied to theorems in Section 4.

Proposition 3.5 *If there exists the only one element z satisfying the condition $(\mathrm{I})_{x,y}$, then z is the greatest lower bound of x and y.*

Proof Let z be the only one element satisfying the condition $(\mathrm{I})_{x,y}$. Suppose that the greatest lower bound u of x and y exists, which is not equal to z, by Proposition 3.1 and Proposition 3.3, then

$$y \wedge x \leqq u, \;\; x \wedge y \leqq u. \tag{3.10}$$

0n the other hand, by the fact that z is the only element with the condition $(\mathrm{I})_{x,y}$, this implies that u doesn't satisfying the condition $(\mathrm{I})_{x,y}$. Then, we have that u doesn't satisfy the inequality (ii),

$$y \wedge x > u \qquad \text{or} \qquad y \wedge x \ngeqq u, \;\; y \wedge x \nleqq u, \tag{3.11}$$

and more u doesn't satisfy the condition (iii), then

$$x \wedge y > u \qquad \text{or} \qquad x \wedge y \ngeqq u, \;\; x \wedge y \ngeqq u. \tag{3.12}$$

The inequality (3.10) contradict with (3.11), (3.12). Hence u doesn't exist. Therefore, by Proposition 3.4, z is the greatest lower bound x and y.

□

For any ordered two elements in a BCK-algebra have several basic properties.

Proposition 3.6 *If $z \leqq x$, then* $z = x \wedge z$.

Proof By the assumption $z \leqq x$ and (7) in Proposition 2.2, we have

$$z = z * 0 = z * (z * x) = x \wedge z.$$

□

Proposition 3.7 *Let $x \wedge y = y \wedge x$. If there exists an element u satisfying the condition $(\mathrm{I})_{x,y}$ and $u \neq x \wedge y$, then $x \wedge y < u$.*

Proof By the assumption, u satisfies the condition $(\mathrm{I})_{x,y}$. By Proposition 3.3, this implies

$$x \wedge y \leqq u. \tag{3.13}$$

By (3.13) and $u \neq x \wedge y$, we have

$$x \wedge y = y \wedge x < u.$$

□

Let $|X|$ denote the cardinal of X for a set X.

Corollary 3.8 *If $x \wedge y = y \wedge x$ and $|I(x,y)| \geqq 2$, then $x \wedge y$ is the least lower element in $I(x,y)$.*

Lemma 3.9 *If x and y satisfy the order $x \leqq y$, and z satisfies the condition* $(\mathrm{I})_{x,y}$, *then $x \wedge y \leqq z = x = y \wedge x$.*

Proof Since z satisfies the condition $(\mathrm{I})_{x,y}$, we have

$$x * z \leqq x * y. \tag{3.14}$$

Further, by $x \leqq y$, we have

$$x * y = 0. \tag{3.15}$$

The inequality (3.14) satisfies (3.15), This implies $x * z \leqq 0$.

Then, we obtain, $x * z = 0$. This implies

$$x \leqq z. \tag{3.16}$$

Conversely, since z is a common lower bound of x and y,

$$x \geqq z. \tag{3.17}$$

Then, by (3.16), (3.17), we have

$$z = x. \tag{3.18}$$

By $x \leqq y$,

$$x = x * (x * y) = y \wedge x. \tag{3.19}$$

On the other hand, by (II) in Definition 2.1, we have

$$x \wedge y = y * (y * x) \leqq x \tag{3.20}$$

From (3.18), (3.19), (3.20),

$$x \wedge y \leqq z = x = y \wedge x.$$

□

Here, the next Proposition 3.10 is derived naturally from Lemma 3.9.

Proposition 3.10 *If x and y satisfy the order $x \leqq y$, then $I(x,y) = \{x\}$.*

When we add the condition $z \leqq x \wedge y$ to the hypothesis of Lemma 3.9, the next Proposition 3.11 holds.

Proposition 3.11 *If x and y satisfy the order $x \leqq y$, z satisfies the condition* $(\mathrm{I})_{x,y}$ *and $z \leqq x \wedge y$, then $z = x \wedge y = y \wedge x$.*

Proposition 3.12 *Let $z \leqq x,\ z \leqq y$. Then $z \wedge x = x \wedge z$ is equivalent to $z \leqq y \wedge x$.*

Proof By $z \leqq x$ and Proposition 3.6, we have $z = x \wedge z$.
Let $x \wedge z = z \wedge x$. Then

$$z \wedge x = x \wedge z = z. \tag{3.21}$$

By $z \leqq y$ and (2) in Proposition 2.2, we have $x * y \leqq x * z$.

Again, using (2) in Proposition 2.2, we have $x * (x * z) \leqq x * (x * y)$.
Then, we obtain

$$z \wedge x \leqq y \wedge x \tag{3.22}$$

Hence, (3.21), (3.22) lead to $z \leqq y \wedge x$.

Conversely, by $z \leqq x$ and Proposition 3.6, we have

$$z = x \wedge z. \tag{3.23}$$

By (3) in Proposition 2.2,

$$\begin{aligned}(z \wedge x) * z &= \{x * (x * z)\} * z \\ &= (x * z) * (x * z) \\ &= 0\end{aligned}$$

Then, we have

$$z \wedge x \leqq z. \tag{3.24}$$

On the other hand, for any y with $z \leqq y$, by the assumption $z \leqq y \wedge x$, we have

$$z * (y \wedge x) = 0.$$

If $y = z$, then $z * (z \wedge x) = 0$, this implies

$$z \leqq z \wedge x. \tag{3.25}$$

Hence, by (3.24), (3.25), we obtain

$$z = z \wedge x. \tag{3.26}$$

Therefore, by (3.23), (3.26), $x \wedge z = z \wedge x$.

□

Using Proposition 3.3 and Proposition 3.12, we have the next theorem. This theorem gives a characterization of the commutativity of any x, y in a BCK-algebra X.

Theorem 3.13 *For x, y, z in a BCK-algebra X, x and y are commutative if and only if there exists an element z satisfying the condition (I)$_{x,y}$ and $z \wedge x = x \wedge z$, $z \wedge y = y \wedge z$.*

We wish to characterize $x \wedge y = y \wedge x$ by using only the condition (I)$_{x,y}$, but the above result is obtained at present.

4 BCK-algebras with Condition (I) and lower semilattices

As are described in (1) of Proposition 2.2, BCK-algebras are partially ordered sets with respect to the relation $\leqq$. Then, we define the next.

Definition 4.1 In a BCK-algebra X, we called that the relation $\leqq$ is *BCK-order*. A BCK-algebra X is called *a lower BCK-semilattice* if when X is a lower semilattice with respect to BCK-order $\leqq$.

We will classify this class further using the condition $(\mathrm{I})_{x,y}$. In the following, let X be a BCK-algebra and any x, y, z in X.

Definition 4.2 If z is the only one element satisfying the condition $(\mathrm{I})_{x,y}$, then we say that z *satisfies the single condition* $(\mathrm{I})_{x,y}$.

Definition 4.3 If z satisfies the following two requirement i) and ii) ;

i) z is the only one element satisfying the condition $(\mathrm{I})_{x,y}$,
ii) $z = x \wedge y = y \wedge x$.

then we say that z *satisfies the canonical condition* $(\mathrm{I})_{x,y}$.

By Definition 4.2 and Definition 4.3, we will define the two special classes in BCK-algebras.

Let X be a BCK-algebra.

Definition 4.4 For any x, y in X, if z exists in X and satisfies the canonical condition $(\mathrm{I})_{x,y}$, then X is called that *a BCK-algebra with Canonical Condition (I)*.

Definition 4.5 For any x, y in X, if z exists in X and satisfies the single condition $(\mathrm{I})_{x,y}$, then X is called that *a BCK-algebra with Condition (I)*.
At this time, we define $x \times y$ by z. Surely, $x \times y = y \times x$.

First, we will show the fundamental properties of BCK-algebras with Canonical Condition (I).

Theorem 4.6 *A commutative BCK-algebra is a BCK-algebra with Canonical Condition (I), and the converse also holds.*

Proof Let X be a commutative BCK-algebra. For any x, y in X, put

$$z = x \wedge y = y \wedge x. \tag{4.1}$$

Clearly, the element z satisfies requirement ii).
And more, by Proposition 3.2, the element z satisfies the condition $(\mathrm{I})_{x,y}$. Here, if any u exists in X and satisfies the condition $(\mathrm{I})_{x,y}$, by (3) in Proposition 3.3, we have

$$y \wedge x \leqq u, x \wedge y \leqq u. \tag{4.2}$$

In addition, since X is a commutative BCK-algebra, we have $x \wedge u = u \wedge x, y \wedge u = u \wedge y$. By Proposition 3.12, this implies

$$u \leqq y \wedge x, \;\; u \leqq x \wedge y. \tag{4.3}$$

Hence, by (4.2), (4.3), implies

$$u = y \wedge x = x \wedge y. \tag{4.4}$$

By (4.1), (4.4), we have $u = z$. Therefore, this shows the requirement i).

The other, the converse is clear. We complete the proof of Theorem 4.6.

□

Theorem 4.7 *A lower semilattice with respect to the operation $\wedge$ is a BCK-algebra with Canonical Condition (I), and the converse also holds.*

Proof Let $X =< X; \wedge, 0 >$ be a lower semilattice for the operation $\wedge$. For any x, y in X, $x \wedge y$ is the greatest lower bound of x and y. Then, put $z = x \wedge y = y \wedge x$. The requirement ii) is clear.

And more, by Proposition 3.1, the greatest lower bound $x \wedge y$ satisfies the condition $(\mathrm{I})_{x,y}$. If any u satisfies the condition $(\mathrm{I})_{x,y}$, by (3) in Proposition 3.3, this implies

$$x \wedge y \leqq u. \tag{4.5}$$

Since X is a lower semilattice for the operation $\wedge$, and $u \leqq x$, we have

$$u = u \wedge x = x \wedge u. \tag{4.6}$$

From (4.6) and Proposition 3.12,

$$u \leqq y \wedge x = x \wedge y. \tag{4.7}$$

Thus, by (4.5), (4.7), $u = x \wedge y$. This shows the requirement i).

Conversely, let X be a BCK-algebra with Canonical Condition (I). For any x, y in X, the element z exists in X and satisfies the requirement i), ii) of Definition 4.2. By Proposition 3.5, we have that the commutative element z is the greatest lower bound of x and y. Therefore, we complete the proof of Theorem 4.7.

□

Clearly, Theorem 4.6 and Theorem 4.7 implies Tanaka's theorem (Theorem 2.3) and the inverse of Theorem 2.3. Consequently, we get the canonical condition $(\mathrm{I})_{x,y}$ in Definition 4.3 is an equivalent to the commutativity in BCK-algebras.

Secondly, we will show the following fundamental property of BCK-algebras with Condition (I).

Theorem 4.8 *A lower semilattice with respect to the operation $\times$ is a BCK-algebra with Condition (I), and the converse also holds.*

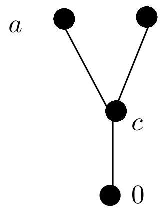
Proof Let $X =< X; \times, 0 >$ be a lower semilattice for the operation $\times$. For any x, y in X, the greatest lower bound of x and y exists in X. This element is defined by $x \times y$ in Definition 4.5. By the assumption, $x \times y$ is the only one element satisfying the condition $(\mathrm{I})_{x,y}$.

Conversely, let X be a BCK-algebra with Condition (I). For any x, y in X, there exists the only one element z in $I(x, y)$. By Proposition 3.5, this implies that z is the greatest lower bound of x and y. Then, we denote $z = x \times y$. The converse is showed. Therefore, we complete the proof of Theorem 4.8.

□

5 Examples of BCK-algebras

Example 5.1 Let $X = \{0, a, b, c\}$ be a set with four elements. And more, the set X satisfies the following Hasse daiagram: $Figure.1$, as BCK-order.



$Figure.1$

At this time, there are four types in the role of algebraic structure of non-isomorphic BCK-algebras on the partially ordered set X: $Figure.1$, as follow four Cayley Tables 1.A, 1.B, 1.C and 1.D exist.

$*$	0	c	a	b
0	0	0	0	0
c	c	0	0	0
a	a	c	0	c
b	b	c	c	0

Table 1.A

$*$	0	c	a	b
0	0	0	0	0
c	c	0	0	0
a	a	c	0	c
b	b	b	b	0

Table 1.B

$*$	0	c	a	b
0	0	0	0	0
c	c	0	0	0
a	a	a	0	c
b	b	c	c	0

Table 1.C

$*$	0	c	a	b
0	0	0	0	0
c	c	0	0	0
a	a	a	0	a
b	b	b	b	0

Table 1.D

The above four BCK-algebras are written as follow $X_A =< X; *, 0 >, X_B =< X; *, 0 >, X_C =< X; *, 0 >$ and $X_D =< X; *, 0 >$ in the order of Cayley Table 1.A, 1.B, 1.C and 1.D.

First, for the BCK-algebra X_A given in Table 1.A, the next Table1.A-$\wedge$ is available for the operation $\wedge$.

$\wedge$	0	c	a	b
0	0	0	0	0
c	0	c	c	c
a	0	c	a	c
b	0	c	c	b

Table 1.A-$\wedge$

Thus, the BCK-algebra X_A is commutative. Further, the operation $\wedge$ matches the lattice-meet $\cap$ obtain from the Hasse diagram: $Figure.1$.

Next, we will examine that the elements satisfying the condition $(\mathrm{I})_{x,y}$ for any two elements in X_A. The results is as follows.

$\{,\}$	$I(x,y)$
$\{0,c\}$	$\{0\}$
$\{0,a\}$	$\{0\}$
$\{0,b\}$	$\{0\}$
$\{c,a\}$	$\{c\}$
$\{c,b\}$	$\{c\}$
$\{a,b\}$	$\{c\}$

Table 1.A-(I)

Therefore, the BCK-algebra X_A is a BCK-algebra with Condition (I). Clearly, the operation$\times$ matches the lattice-meet $\cap$, and this algebra also coincides with the operation $\wedge$. Then, the BCK-algebra X_A is a BCK-algebra with Canonical Condition (I).

Second, for the BCK-algebra X_B given in the Table 1.B, the following relation holds for the operation $\wedge$.

$$c = c * (c * b) = b \wedge c \neq c \wedge b = b * (b * c) = 0,$$

$$c = a * (a * b) = b \wedge a \neq a \wedge b = b * (b * a) = 0.$$

Therefore, the BCK-algebra X_B is not commutative. However, we will examine that the elements satisfying the condition $(\mathrm{I})_{x,y}$ for any two elements in X_B. The result is as follows.

$\{,\}$	$I(x,y)$
$\{0,c\}$	$\{0\}$
$\{0,a\}$	$\{0\}$
$\{0,b\}$	$\{0\}$
$\{c,a\}$	$\{c\}$
$\{c,b\}$	$\{c\}$
$\{a,b\}$	$\{c\}$

Table 1.B-(I)

From this table, any two elements in the BCK-algebra X_B satisfy the single condition $(\mathrm{I})_{x,y}$. Then, the BCK-algebra X_B is a BCK-algebra with Condition (I). Therefore, the following the Table 1.B-$\times$ is obtained with respect to the operation $\times$.

$\times$	0	c	a	b
0	0	0	0	0
c	0	c	c	c
a	0	c	a	c
b	0	c	c	b

Table 1.B-$\times$

This table matches the lattice-meet $\cap$ that obtained from the Hasse diagram: $Figure.1$.

Similarly, for the BCK-algebra X_C is given in the Table 1.C, the following relations hold for the operation $\wedge$.

$$c = c * (c * a) = a \wedge c \neq c \wedge a = a * (a * c) = 0,$$

$$c = b * (b * a) = a \wedge b \neq b \wedge a = a * (a * b) = a.$$

Therefore, the BCK-algebra X_C is not commutative. However, we will examine that the elements satisfying the condition $(\mathrm{I})_{x,y}$ for any two elements in X_C. The result is follows.

$\{,\}$	$I(x,y)$
$\{0,c\}$	$\{0\}$
$\{0,a\}$	$\{0\}$
$\{0,b\}$	$\{0\}$
$\{c,a\}$	$\{c\}$
$\{c,b\}$	$\{c\}$
$\{a,y\}$	$\{c\}$

Table 1.C-(I)

From this table, X_C is a BCK-algebra with Condition (I). This table is consistent with the lattice-meet $\cap$ obtained from the Hasse diagram: $Figure.1$.

Thirdly, for the operation $\wedge$ in the BCK-algebra X_D, we have the following relation.

$$0 = a * (a * c) = c \wedge a \neq a \wedge c = c * (c * a) = c,$$

$$0 = b * (b * c) = c \wedge b \neq b \wedge c = c * (c * b) = c$$

Then, the BCK-algebra X_D is not commutative. Further, we will examine that the elements satisfying the condition $(\mathrm{I})_{x,y}$ for any two elements in X_D, the following is obtained.

$\{,\}$	$I(x,y)$
$\{0,c\}$	$\{0\}$
$\{0,a\}$	$\{0\}$
$\{0,b\}$	$\{0\}$
$\{c,a\}$	$\{c\}$
$\{c,b\}$	$\{c\}$
$\{a,b\}$	$\{0,c\}$

Table 1.D-(I)

The following can be understood. In the BCK-algebra X_D, there are two elements satisfying the condition $(\mathrm{I})_{x,y}$ for a and b.

$$a = a * 0 \leqq a * b = a, \;\; b = b * 0 \leqq b * a = b,$$

$a = a * c \leqq a * b = b,\ b = b * c \leqq b * a = b.$

Therefore, two elements a and b don't satisfy the single condition $(\mathrm{I})_{x,y}$, then the BCK-algebra X_D is not a BCK-algebra with Condition (I).

Example 5.2 Let $Y = \{0, x, y, 1\}$ be a set with four elements. And more, the set Y satisfies the following totally order $0 \leqq a \leqq b \leqq 1$, as BCK-order of Y.

At this time, there are six types in the role of algebraic structure of non-isomorfic BCK-algebras on the totally ordered set Y, as follow Cayley Table 2.A, 2.B, 2.C, 2.D, 2.E and 2.F exist.

$*$	0	a	b	1
0	0	0	0	0
a	a	0	0	0
b	b	a	0	0
1	1	b	a	0

Table 2.A

$*$	0	a	b	1
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
1	1	b	a	0

Table 2.B

$*$	0	a	b	1
0	0	0	0	0
a	a	0	0	0
b	b	a	0	0
1	1	a	a	0

Table 2.C

$*$	0	a	b	1
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
1	1	1	b	0

Table 2.D

$*$	0	a	b	1
0	0	0	0	0
a	a	0	0	0
b	b	a	0	0
1	1	1	1	0

Table 2.E

$*$	0	a	b	1
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
1	1	1	1	1

Table 2.F

The above six BCK-algebras are written as follow $Y_A =< Y; *, 0 >, Y_B =< Y; *, 0 >, Y_C =< Y; *, 0 >, Y_D =< Y; *, 0 >, Y_E =< Y; *, 0 >$ *and* $Y_F =< Y; *, 0 >$ in order of Cayley Table 2.A, 2.B, 2.C, 2.D, and 2.F.

First, for the BCK-algebra Y_A given in Table 2.A, the following Table 2. A-$\wedge$ is available for the operation $\wedge$.

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

Table 2.A-$\wedge$

Therefore, the BCK-algebra Y_A is commutative. Further, the operation $\wedge$ matches the lattice-meet $\cap$ obtain from BCK-order of Y.

Next, we will examine that the elements satisfying the condition $(\mathrm{I})_{x,y}$ for any two elements in Y_A. The result is the following.

$\{,\}$	$I(x,y)$
$\{0,a\}$	$\{0\}$
$\{0,b\}$	$\{0\}$
$\{0,1\}$	$\{0\}$
$\{a,b\}$	$\{a\}$
$\{a,1\}$	$\{a\}$
$\{b,1\}$	$\{b\}$

Table 2.A-(I)

Therefore, the BCK-algebra Y_A is a BCK-algebra with Condition (I). Clearly, the operation $\times$ matches the lattice-meet $\cap$, and this algebra also coincides with the operation $\wedge$. Then, the BCK-algebra Y_A is a BCK-algebra with Canonical Condition (I).

Second, for the BCK-algebras Y_B, Y_C, Y_D, Y_E, and Y_F given in the Table 2.B, Table 2.C, Table 2.D, Table 2.E and Table 2.F, the following relation holds for the operation $\wedge$.

In the BCK-algebra Y_B,

$$0 = b * (b * a) = a \wedge b \neq b \wedge a = a * (a * b) = a \ ;$$

In the BCK-algebra Y_C,

$$a = 1 * (1 * b) = b \wedge 1 \neq 1 \wedge b = b * (b * 1) = b \ ;$$

In the BCK-algebra Y_D,

$$0 = b * (b * a) = a \wedge b \neq b \wedge a = a * (a * b) = a,$$

$$0 = 1 * (1 * a) = x \wedge 1 \neq 1 \wedge a = a * (a * 1) = b \ ;$$

In the BCK-algebra Y_E,

$$0 = 1 * (1 * a) = a \wedge 1 \neq 1 \wedge a = a * (a * 1) = a,$$

$$0 = 1 * (1 * b) = b \wedge 1 \neq 1 \wedge b = b * (b * 1) = b \ ;$$

In the BCK-algebra Y_F,

$$0 = b * (b * a) = a \wedge b \neq b \wedge a = a * (a * b) = a,$$

$$0 = 1 * (1 * a) = a \wedge 1 \neq 1 \wedge a = a * (*1) = a.$$

$$0 = 1 * (1 * b) = b \wedge 1 \neq 1 \wedge b = b * (b * 1) = b \ ;$$

Therefore, the BCK-algebras Y_B, Y_C, Y_D, Y_E and Y_F are not commutative.

Next, we consider that the elements satisfying the condition $(\mathrm{I})_{x,y}$ for any two elements in Y_B. This is clear from the next Corollary 5.3. Because of that the set Y is a totally ordered set, the Proposition 3.10 implies the Corollary 5.3.

Corollary 5.3 *For the totally ordered sets, the algebraic structure of the BCK-algebra with Condition (I) is always given.*

Then, the BCK-algebras Y_B, Y_C, Y_D, Y_E, and Y_F are BCK-algebras with Condition (I).

6 Additional Remarks

P. M. Idziak showed that a lower BCK-semilattice is as a variety in the article [1]. Then, a BCK-algebra with Condition (I) should be as a variety. Therefore, we should be able to define this class with the condition for only identities including the operation $\times$. However, this condition expression has not been provided yet. If we can this condition expression, we will investigate into BCK-algebras with Condition (I) more deeply.

In addition, we don't know whether the BCK-algebras with Condition (I) is the maximum class in lower BCK-semilattices.

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Mino Jiyu Gakuen Senior High School, Miyayama 4-21-1, Toyonaka, Osaka, Japan.
E-mail: kumazawamasaaki0717@gmail.com