L^p-boundedness of a Hausdorff operator associated with change of variables and weights

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Abstract

Multivariate Hausdorff operators associated with linear transformations on $L^p(\mathbb{R}^n)$ are investigated by Brown and Moricz. We replace the linear transformation with a general change of variables and introduce modified Hausdorff operators $\mathcal{H}_\psi$ associated with a change of variables and weights. We obtain a condition of ψ under which the operator is bounded from L^p to L^p. The modified Hausdorff operators cover the Hausdorff operators defined on the Euclidean space, the Dunkl hypergroup and the Jacobi hypergroup. In each case, we give conditions of ψ under which the operators are bounded from L^p to L^p.

1 The modified Hausdorff operator

Let $\mu(t)$ be a Borel measure on $\mathbb{R}^n$ and $A(t)$ a $n \times n$ matrix whose entries $a_{ij}(t)$ are functions on $\mathbb{R}^n$. Brown and Moricz [2] introduce the multivariate Hausdorff operator H_ψ acting on functions on $\mathbb{R}^n$ as

$$H_\psi(f)(x) = \int_{\mathbb{R}^n} \psi(t) f(A(t)x) d\mu(t)$$

provided that the integral on the right-hand side exists. For $1 \leq p \leq \infty$ they obtain a condition of ψ under which H_ψ is bounded from L^p to L^p (see §3.1). Moreover, the boundedness on H^p, BMO, Herz-type spaces, Morrey-type spaces, and so on are investigated by many authors (see [1]

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and references therein). The Hausdorff operators are generalized on abstract groups. For example, on the Heisenberg groups, Guo, Sun and Zhao [4] obtain the sharp L^p estimates of high-dimensional and multilinear Hausdorff operators. Then the operators on other function spaces are investigated (see [7] and references therein). In this paper we introduce a modified Hausdorff operator $\mathcal{H}_\psi$ by replacing $A(t)x$ with a general change of variable $F_t(x)$ and $d\mu(t)$ with a weight function $\omega(t)dt$. In particular, treating the cases that the weight functions $\omega(t)$ corresponds to the Dunkl and the Jacobi hypergroups respectively, we can obtain some conditions of ψ under which $\mathcal{H}_\psi$ for the Dunkl and the Jacobi hypergroups are bounded from L^p to L^p (see §3.2 and §3.3).

Let $U \subset \mathbb{R}^n$ be an open subset and let $F : U \to \mathbb{R}^n$ be a C^1 function. We suppose that F is one-to-one and that, for all $x \in U$, the derivative $DF(x)$ is invertible. Hence $V = F(U) \subset \mathbb{R}^n$ is open and $F : U \to V$ is a diffeomorphism. Then for a suitable function f on V,

$$\int_V f(v)dv = \int_U f(F(u))|\det DF(u)|du,$$

where dv and du are Lebesgue measures on $\mathbb{R}^n$. Let ω_U and ω_V are positive functions on U and V respectively. We denote by $L^p(U, \omega_U)$ (resp. $L^p(V, \omega_V)$) the space of L^p functions on U with respect to $\omega_U(u)du$ (resp. on V with respect to $\omega_V(v)dv$). For $g \in L^p(U, \omega_U)$, we put

$$g^*_F(v) = g(F^{-1}(v))\frac{\omega_U(F^{-1}(v))}{\omega_V(v)}|\det DF(F^{-1}(v))|^{-1}.$$

If $g \in L^1(U, \omega_U)$, then it follows from the change of variables formula that

$$\int_V g^*_F(v)\omega_V(v)dv = \int_U g(u)\omega_U(u)du. \tag{1}$$

We now suppose that F depends on a parameter $t \in U$, and write $F = F_t$. Let ψ be a positive function on U. We define the Hausdorff operator $\mathcal{H}_\psi$ acting on functions on V and its dual $\mathcal{H}^*_\psi$ acting on functions on U as follows.

$$(\mathcal{H}_\psi f)(u) = \int_U \psi(t)f(F_t(u))\omega_U(t)dt,$$
$$(\mathcal{H}^*_\psi g)(v) = \int_U \psi(t)g^*_{F_t}(v)\omega_U(t)dt.$$

Actually, they satisfy the following relation.

$$
\begin{aligned}
&\int_U (\mathcal{H}_\psi f)(u)\overline{g(u)}\omega_U(u)du \\
&= \int_U \psi(t)\omega_U(t)(\int_U f(F_t(u))\overline{g(u)}\omega_U(u)du)dt \\
&= \int_U \psi(t)\omega_U(t)(\int_V f(v)\overline{g(F_t^{-1}(v))}\omega_U(F_t^{-1}(v))|\det DF_t(F_t^{-1}(v))|^{-1}dv)dt \\
&= \int_V f(v)(\int_U \psi(t)\overline{g^*_{F_t}(v)}\omega_U(t)dt)\omega_V(v)dv \qquad (2)\\
&= \int_V f(v)\overline{(\mathcal{H}^*_\psi g)(v)}\omega_V(v)dv.
\end{aligned}
$$

Lemma 1.1. *We suppose that* $\psi \in L^1(U,\omega_U)$. *Then for all* f *in* $L^\infty(V,\omega_V)$,

$$\|\mathcal{H}_\psi f\|_{L^\infty(U,\omega_U)} \le \|\psi\|_{L^1(U,\omega_U)}\|f\|_{L^\infty(V,\omega_V)}.$$

Proof. This is obvious from the definition of $\mathcal{H}_\psi$. □

Lemma 1.2. *We suppose that*

$$d_\psi = \sup_{v\in V}\int_U \psi(t)\frac{\omega_U(F_t^{-1}(v))}{\omega_V(v)}|\det DF_t(F_t^{-1}(v))|^{-1}\omega_U(t)dt < \infty. \qquad (3)$$

Then for all f *in* $L^1(V,\omega_V)$,

$$\|\mathcal{H}_\psi f\|_{L^1(U,\omega_U)} \le d_\psi\|f\|_{L^1(V,\omega_V)}.$$

Proof. By letting $g = 1$, the inequality follows from (2). □

Therefore, by using the interpolation and the duality, we can deduce the following.

Theorem 1.3. *Let* $1 \le p \le \infty$ *and* $\frac{1}{p}+\frac{1}{p^*}=1$. *Then for all* f *in* $L^p(V,\omega_V)$ *and all* g *in* $L^p(U,\omega_U)$,

$$
\begin{aligned}
\|\mathcal{H}_\psi f\|_{L^p(U,\omega_U)} &\le (d_\psi)^{\frac{1}{p}}\|\psi\|^{\frac{1}{p^*}}_{L^1(U,\omega_U)}\|f\|_{L^p(V,\omega_V)},\\
\|\mathcal{H}^*_\psi g\|_{L^p(V,\omega_V)} &\le (d_\psi)^{\frac{1}{p^*}}\|\psi\|^{\frac{1}{p}}_{L^1(U,\omega_U)}\|g\|_{L^p(U,\omega_U)}.
\end{aligned}
$$

2 Another L^p boundedness

To obtain the L^p boundedness of $\mathcal{H}_\psi$ in Theorem 1.3 we use the interpolation. Here we shall calculate the L^p norm of $\mathcal{H}_\psi$ directly. We put

$$\rho(t) = \inf_{u \in U} \frac{|\det DF_t(u)|\omega_V(F_t(u))}{\omega_U(u)}$$

and for $1 \le p \le \infty$,

$$C^p_{\psi,\rho} = \int_U \psi(t)\rho(t)^{-\frac{1}{p}}\omega_U(t)dt. \tag{4}$$

Theorem 2.1. *Let $1 \le p \le \infty$ and $\frac{1}{p} + \frac{1}{p^*} = 1$. Then for all f in $L^p(V, \omega_V)$ and all g in $L^p(U, \omega_U)$,*

$$\|\mathcal{H}_\psi f\|_{L^p(U,\omega_U)} \le C^p_{\psi,\rho}\|f\|_{L^p(V,\omega_V)},$$
$$\|\mathcal{H}^*_\psi g\|_{L^p(V,\omega_V)} \le C^{p^*}_{\psi,\rho}\|g\|_{L^p(U,\omega_U)}$$

provided that $C^p_{\psi,\rho} < \infty$ and $C^{p^*}_{\psi,\rho} < \infty$ respectively.

Proof. We shall prove the second inequality. Then for $g \in L^p(U, \omega_U)$ and $1 \le p < \infty$, we see that

$$\begin{aligned}\|g^*_{F_t}\|^p_{L^p(V,\omega_V)} &= \int_U |g(u)|^p\Big(\frac{|\det DF_t(u)|\omega_V(F_t(u))}{\omega_U(u)}\Big)^{1-p}\omega_U(u)du \\ &\le \rho(t)^{1-p}\|g\|^p_{L^p(U,\omega_U)}.\end{aligned} \tag{5}$$

Hence, by the definition of $\mathcal{H}^*_\psi g$ and (5), we see that

$$\begin{aligned}\|\mathcal{H}^*_\psi g\|_{L^p(V,\omega_V)} &\le \int_U \psi(t)\omega_U(t)\|g^*_{F_t}\|_{L^p(V,\omega_V)}dt \\ &\le \int_U \psi(t)\rho(t)^{\frac{1}{p}-1}\omega_U(t)dt \cdot \|g\|_{L^p(U,\omega_U)}.\end{aligned}$$

The case $p = \infty$ is obvious. The first inequality follows by the duality. Here we give a direct proof. We suppose $p < \infty$. We see that

$$\begin{aligned}&\|\mathcal{H}_\psi f\|_{L^p(U,\omega_U)} \\ &\le \int_U \psi(t)\omega_U(t)\|f(F_t(\cdot))\|_{L^p(U,\omega_U)}dt \\ &\le \int_U \psi(t)\omega_U(t)\Big(\int_V |f(v)|^p\Big(\frac{|\det DF_t(F_t^{-1}(v))|\omega_V(v)}{\omega_U(F_t^{-1}(v))}\Big)^{-1}\omega_V(v)dv\Big)^{\frac{1}{p}}dt \\ &\le \int_U \psi(t)\rho(t)^{-\frac{1}{p}}\omega_U(t)dt \cdot \|f\|_{L^p(V,\omega_V)}.\end{aligned} \tag{6}$$

The case $p = \infty$ is obvious. □

Remark 2.2. We note that $d_\psi \leq C^1_{\psi,\rho}$. Moreover, by using the Hölder inequality, it follows that $C^p_{\psi,\rho} \leq (C^1_{\psi,\rho})^{\frac{1}{p}} \|\psi\|^{\frac{1}{p*}}_{L^1(U,\omega_U)}$ and thus, $C^1_{\psi,\rho} \geq (C^p_{\psi,\rho})^p \|\psi\|^{1-p}_{L^1(U,\omega_U)}$. Therefore, if

$$\frac{|\det DF_t(u)|\omega_V(F_t(u))}{\omega_U(u)}$$

does not depend on u (see §3), then it follows that $d_\psi = C^1_{\psi,\rho}$ and thus,

$$(d_\psi)^{\frac{1}{p}} \|\psi\|^{\frac{1}{p^*}}_{L^1(U,\omega_U)} \geq C^p_{\psi,\rho}.$$

3 Variants of weights

Our modified Hausdorff operator $\mathcal{H}_\psi$ depends on a weight function ω_U. Therefore, by changing the weight, we can treat the Hausdorff operators for the Euclidean space, the Dunkl hypergroup, and the Jacobi hypergroup similarly

3.1 Euclidean space

Let $A(u) = (a_{ij}(u))^n_{i,j=1}$ be an $n \times n$ matrix, where coefficients $a_{ij}(u)$ are measurable functions of u and $A(u)$ may be singular on a set of measure zero. We take $U = V = \mathbb{R}^n$,

$$\begin{array}{cccc} F_t : & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & \Cup & & \Cup \\ & x & \longmapsto & xA(t), \end{array}$$

and $w_U(x) = w_V(x) = 1$. Here $xA(t)$ is the multiplication of the row vector x and the matrix $A(t)$. Then

$$g^*_{F_t}(x) = g\left(xA^{-1}(t)\right)|\det(A(t))|^{-1}.$$

Hence the Hausdorff type operator and its dual are given as follows.

$$(\mathcal{H}_\psi f)(u) = \int_{\mathbb{R}^n} \psi(t) f(uA(t))dt,$$

$$\left(\mathcal{H}^*_\psi g\right)(v) = \int_{\mathbb{R}^n} \psi(t) g(vA^{-1}(t))|\det(A(t))|^{-1}dt.$$

Moreover, it follows that

$$\rho(t) = \inf_{u \in U} \frac{|\det DF_t(u)|\omega_V(F_t(u))}{\omega_U(u)} = |\det A(t)|.$$

Then the following corollary is obtained (see [2]).

Corollary 3.1. *Let $A(t) = (a_{ij}(t))$ be an $n \times n$ matrix and may be singular on a set of measure zero in $\mathbb{R}^n$. Let $1 \le p \le \infty$ and $\frac{1}{p} + \frac{1}{p^*} = 1$. We put*

$$C^p_{\psi,A} = \int_{\mathbb{R}^n} \psi(t) |\det A(t)|^{-\frac{1}{p}} dt.$$

Then for all f in $L^p(\mathbb{R}^n)$,

$$\|\mathcal{H}_\psi f\|_{L^p(\mathbb{R}^n)} \le C^p_{\psi,A} \|f\|_{L^p(\mathbb{R}^n)},$$
$$\|\mathcal{H}^*_\psi f\|_{L^p(\mathbb{R}^n)} \le C^{p^*}_{\psi,A} \|f\|_{L^p(\mathbb{R}^n)}$$

provided that $C^p_{\psi,A} < \infty$ and $C^{p^*}_{\psi,A} < \infty$ respectively.

Remark 3.2. Let $A(t)$ be a diagonal matrix $\mathrm{diag}(t_1, t_2, \cdots, t_n)$. Then $g^*_{F_t}(x)$ is a kind of dilation of g. Actually, when $n = 1$, $g^*_{F_t}(x)$ coincides with the dilation of g and $\mathcal{H}_\psi$ is the classical one-dimensional Hausdorff operator. However, there are various kinds of extension of the classical Hausforff operators. For example, in [5], the case that $U = V = \mathbb{R}^n$, $\omega_U(x) = \omega_V(x) = \|x\|^\alpha$ and $F_t(x) = \frac{x}{\|t\|}$ is investigated.

3.2 Dunkl hypergroup

As an extension of one-dimensional Hausdorff operator, we shall consider a modified Hausdorff operator related with the Dunkl hypergroup (see [3]). Let $\kappa = (\kappa_1, \cdots, \kappa_n)$ where each κ_j is a nonnegative real number. Let $d\mu_\kappa$ denote the associated measure given for $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ by

$$d\mu_\kappa(x) = h^2_\kappa(x) dx,$$

where h_κ is the $\mathbb{Z}^n_2$-invariant function defined by

$$h_\kappa(x) = \prod_{j=1}^{n} |x_j|^{\kappa_j}.$$

Let $A(s) = \mathrm{diag}(a_1(s), \cdots, a_n(s))$ be a diagonal matrix, where coefficients $a_j(s)$ are measurable functions of s and $A(s)$ may be singular on a set of measure zero. We take $U = V = \mathbb{R}^n$,

$$\begin{array}{cccc} F_t: & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & \Cup & & \Cup \\ & x & \longmapsto & xA(t), \end{array}$$

and $w_U(x) = w_V(x) = h_\kappa^2(x)$. Then

$$g_{F_t}^*(x) = g(xA^{-1}(t))\frac{|\det(A(t))|^{-1}}{\prod_{j=1}^n |a_j(t)|^{2\kappa_j}} = g(xA^{-1}(t))\frac{1}{\prod_{j=1}^n |a_j(t)|^{2\kappa_j+1}}.$$

Hence the modified Hausdorff operator and its dual are given as follows.

$$(\mathcal{H}_{\kappa,\psi}f)(x) = \int_{\mathbb{R}^n} \psi(s)f(xA(s))d\mu_\kappa(s),$$
$$(\mathcal{H}_{\kappa,\psi}^*g)(v) = \int_{\mathbb{R}^n} \psi(t)g(vA^{-1}(t))\frac{d\mu_\kappa(t)}{\prod_{j=1}^n |a_j(t)|^{2\kappa_j+1}}.$$

Moreover, it follows that

$$\rho(t) = \inf_{u\in U} \frac{|\det DF_t(u)|\omega_V(F_t(u))}{\omega_U(u)} = \prod_{j=1}^n |a_j(t)|^{2\kappa_j+1}.$$

Corollary 3.3. *Let $A(t) = \mathrm{diag}(a_1(t), \cdots, a_n(t))$ be a diagonal matrix and may be singular on a set of measure zero in $\mathbb{R}^n$. Let $1 \le p \le \infty$ and $\frac{1}{p} + \frac{1}{p^*} = 1$. We put*

$$C_{\psi,A,\kappa}^p = \int_{\mathbb{R}^d} \psi(t)\Big(\prod_{j=1}^n |a_j(t)|^{2\kappa_j+1}\Big)^{-\frac{1}{p}} d\mu_\kappa(t).$$

Then for all f in $L^p(\mathbb{R}^n, d\mu_\kappa)$,

$$\|\mathcal{H}_{\kappa,\psi}f\|_{L^p(\mathbb{R}^n,d\mu_\kappa)} \le C_{\psi,A,\kappa}^p \|f\|_{L^p(\mathbb{R}^n,d\mu_\kappa)},$$
$$\|\mathcal{H}_{\kappa,\psi}^*f\|_{L^p(\mathbb{R}^n,d\mu_\kappa)} \le C_{\psi,A,\kappa}^{p^*} \|f\|_{L^p(\mathbb{R}^n,d\mu_\kappa)}$$

provided that $C_{\psi,A,\kappa}^p < \infty$ and $C_{\psi,A,\kappa}^{p^*} < \infty$ respectively.

Next let us consider the case that $A(s) = (a_{ij}(s))$ is a non-singular upper triangular matrix with $a_{ij}(s) \ge 0$ for all $j \ge i$. Then for $u = (x_1, x_2, \cdots, x_n)$,

$$\frac{|\det DF_t(u)|\omega_V(F_t(u))}{\omega_U(u)} = |\det(A(t)|\frac{\prod_{j=1}^n |\sum_{i=1}^j a_{ij}(t)x_i|^{2\kappa_j}}{\prod_{j=1}^d |x_j|^{2\kappa_j}}$$
$$= |\det(A(t)|\prod_{j=1}^n |a_{jj}(t) + \sum_{i<j} a_{ij}(t)\frac{x_i}{x_j}|^{2\kappa_j}.$$

Hence, by taking the infimum of the above over $u \in \mathbb{R}_+^d$, then the infimum $\rho(t)$ is given by $\prod_{j=1}^n |a_{jj}(t)|^{2\kappa_j+1}$. Moreover, $\{xA(t) \mid x \in \mathbb{R}_+^n\} \subset \mathbb{R}_+^n$. Then, noting (5) and (6), we can obtain the following.

Corollary 3.4. *Let $A(s) = (a_{ij}(s))$ be a non-singular upper triangular matrix with $a_{ij}(s) \geq 0$ for all $j \geq i$. Let $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{p^*} = 1$. Then for all f in $L^p(\mathbb{R}^n_+, d\mu_\kappa)$,*

$$\|\mathcal{H}_{\kappa,\psi} f\|_{L^p(\mathbb{R}^n, d\mu_\kappa)} \leq C^p_{\psi,A,\kappa} \|f\|_{L^p(\mathbb{R}^n_+, d\mu_\kappa)},$$
$$\|\mathcal{H}^*_{\kappa,\psi} f\|_{L^p(\mathbb{R}^n, d\mu_\kappa)} \leq C^{p^*}_{\psi,A,\kappa} \|f\|_{L^p(\mathbb{R}^n_+, d\mu_\kappa)}$$

provided that $C^p_{\psi,A,\kappa} < \infty$ and $C^{p^*}_{\psi,A,\kappa} < \infty$ respectively.

3.3 Jacobi hypergroup

We shall consider a modified Hausdorff operator related with the Jacobi hypergroup $(\mathbb{R}_+, *, \Delta)$ (see [6]). Let $\alpha \geq \beta \geq -\frac{1}{2}$, $(\alpha, \beta) \neq (\frac{1}{2}, \frac{1}{2})$ and put $\Delta(x) = (\sinh x)^{2\alpha+1}(\cosh x)^{2\beta+1}$ on $\mathbb{R}_+$. We define the L^p-norm of a function f on $\mathbb{R}_+$ by

$$\|f\|_{L^p(\Delta)} = \Big(\int_0^\infty |f(x)| \Delta(x) dx \Big)^{\frac{1}{p}}.$$

Let $L^p(\Delta)$ denote the space of functions on $\mathbb{R}_+$ with finite L^p-norm. For $\phi \in L^1(\Delta)$ we define the dilation $\phi_t, t > 0$ of ϕ as

$$\phi_t(x) = \frac{1}{t} \frac{1}{\Delta(x)} \phi\left(\frac{x}{t}\right) \Delta\left(\frac{x}{t}\right).$$

for $x \in \mathbb{R}_+$. We see that $\|\phi_t\|_{L^1(\Delta)} = \|\phi\|_{L^1(\Delta)}$. We take $U = V = \mathbb{R}_+$,

$$\begin{array}{cccc} F_t: & \mathbb{R}_+ & \longrightarrow & \mathbb{R}_+ \\ & \Cup & & \Cup \\ & x & \longmapsto & xt, \end{array}$$

and $w_U(x) = w_V(x) = \Delta(x)$. Then $L^1(U, \omega_U) = L^1(V, \omega_V) = L^1(\Delta)$ and

$$\begin{aligned} g^*_{F_t}(x) =& g(F_t^{-1}(x)) \frac{\omega_U(F_t^{-1}(x))}{\omega_V(x)} |\det DF(F_t^{-1}(x))|^{-1} \\ =& \frac{1}{t} \frac{1}{\Delta(x)} g\left(\frac{x}{t}\right) \Delta\left(\frac{x}{t}\right) = g_t(x). \end{aligned}$$

Hence the modified Hausdorff operator and its dual are given as follows.

$$(\mathcal{H}_\psi f)(u) = \int_0^\infty f(ut) \psi(t) \Delta(t) dt,$$
$$(\mathcal{H}^*_\psi g)(v) = \int_0^\infty g_t(v) \psi(t) \Delta(t) dt.$$

Corollary 3.5. *We suppose that $\psi \in L^1(\Delta)$. Then for all $f \in L^\infty(\Delta)$,*

$$\|\mathcal{H}_\psi f\|_{L^\infty(\Delta)} \leq \|\psi\|_{L^1(\Delta)} \|f\|_{L^\infty(\Delta)}$$

and for all $g \in L^1(\Delta)$,

$$\|\mathcal{H}^*_\psi g\|_{L^1(\Delta)} \leq \|\psi\|_{L^1(\Delta)} \|g\|_{L^1(\Delta)}.$$

We note that if $t < 1$, then

$$\rho(t) = \inf_{0 \leq u < \infty} \frac{t\Delta(tu)}{\Delta(u)} = 0$$

and if $t \geq 1$, then $\rho(t) = t^{2\alpha+2}$, because $t \sinh u \leq \sinh(tu)$. Therefore, if $\psi \in L^1(\Delta)$ is supported on $[1, \infty)$, then $C^p_{\psi,\rho}$ equals

$$C^p_{\psi,\Delta} = \int_1^\infty \psi(t) t^{-\frac{2\alpha+2}{p}} \Delta(t) dt \leq \|\psi\|_{L^1(\Delta)}$$

and also, $C^{p^*}_{\psi,\Delta} \leq \|\psi\|_{L^1(\Delta)}$ for $\frac{1}{p} + \frac{1}{p^*} = 1$. Therefore, we can obtain the following.

Corollary 3.6. *Let $1 \leq p \leq \infty$. We suppose that $\psi \in L^1(\Delta)$ and is supported on $[1, \infty)$. Then for all f in $L^p(\Delta)$,*

$$\|\mathcal{H}_\psi f\|_{L^p(\Delta)} \leq C^p_{\psi,\Delta} \|f\|_{L^p(\Delta)},$$
$$\|\mathcal{H}^*_\psi f\|_{L^p(\Delta)} \leq C^{p^*}_{\psi,\Delta} \|f\|_{L^p(\Delta)}.$$

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