Log-majorization on spectral geometric mean and relative entropy

Dedicated to the memory of Professor Zirô Takeda with sincere respect

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Abstract. In this paper, we prove that the spectral geometric mean $A\,\mathrm{sp}_\alpha B$ for positive definite matrices A and B is log-majorized by $A^{\frac{1-\alpha}{2}}B^\alpha A^{\frac{1-\alpha}{2}}$ for $-1 \le \alpha \le 0$. Precisely, it is a new result for $-1 \le \alpha \le -\frac{1}{2}$, and we give an alternative simple proof for $-\frac{1}{2} \le \alpha \le 0$. Accordingly we discuss some applications to relative entropies of Tsallis type.

1 Introduction. Throughout this note, we denote by $A \ge 0$ if a matrix A is positive semidefinite and $A > 0$ if A is positive definite. For a fixed $\alpha \in [0,1]$, α-spectral geometric mean is defined by

$$A\ \mathrm{sp}_\alpha\ B = (A^{-1}\ \#\ B)^\alpha A (A^{-1}\ \#\ B)^\alpha$$

for $A, B > 0$, where $A \# B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}$, the operator geometric mean of A and B.

Now, the log–majorization $X \prec_{\log} Y$ for $X, Y \ge 0$ means that

$$\prod_{j=1}^{k} \lambda_j(X) \le \prod_{j=1}^{k} \lambda_j(Y),\ \ (k = 1, \cdots, n-1)$$

and $\det X = \det Y$, where $\lambda_1(X) \ge \cdots \ge \lambda_n(X)$ are the eigenvalues of X in decreasing order. Recently Furuichi-Seo proposed the following conjecture on spectral geometric mean in [7]. Precisely they said: We do not know whether it holds or not for $(-1, -\frac{1}{2})$. But they implicitly did it, we suppose. As a matter of fact, they give a proof for the case $\alpha = -1$ [7, Theorem 4.6].

Conjecture FS. If $-1 \le \alpha \le -\dfrac{1}{2}$, then

$$A\ \mathrm{sp}_\alpha B \prec_{\log} A^{\frac{1-\alpha}{2}} B^\alpha A^{\frac{1-\alpha}{2}}$$

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holds for $A, B > 0$.

In this note, we prove it affirmatively. Related to this, it is shown in the same paper [7, Theorem 4.3] that it holds for $-\frac{1}{2} \le \alpha < 0$. We give a simple proof of it depending on the Löwner-Heinz inequality only. As an application, we discuss on relative entropies of Tsallis type.

2 Proof of the conjecture. The main tool of the proof is a reverse inequality of the BLP-inequality cited in [11] and [12]. We cite it for convenience:

(RBLP) If $A, B > 0$, then

$$\|A^{\frac{1+t}{2}} B^t A^{\frac{1+t}{2}}\| \ge \|A^{\frac{1}{2}} (A^{\frac{s}{2}} B^s A^{\frac{s}{2}})^{\frac{t}{s}} A^{\frac{1}{2}}\|$$

holds for $t \ge s \ge 1$.

In our discussion below, it is useful for $s = 1$. That is,

(RBLP-1) If $A, B > 0$, then

$$\|A^{\frac{1+t}{2}} B^t A^{\frac{1+t}{2}}\| \ge \|A^{\frac{1}{2}} (A^{\frac{1}{2}} B A^{\frac{1}{2}})^t A^{\frac{1}{2}}\|$$

holds for $t \ge 1$.

We here cite the Löwner-Heinz inequality just to be sure:

(LH) $$A \ge B \ge 0 \;\Rightarrow\; A^s \ge B^s \quad \text{for } s \in [0, 1].$$

In addition, we use implicitly its tarnspose $A^{\frac{1}{2}}(A \# B^{-1})^{2\alpha} A^{\frac{1}{2}}$ instead of the spectral geometric mean $A \text{ sp}_\alpha\ B = (A^{-1} \# B)^\alpha A (A^{-1} \# B)^\alpha$.

Theorem 2.1. *If* $-1 \le \alpha \le -\dfrac{1}{2}$, *then*

$$A \text{ sp}_\alpha B \prec_{\log} A^{\frac{1-\alpha}{2}} B^\alpha A^{\frac{1-\alpha}{2}}$$

holds for $A, B > 0$.

Proof. It is easily checked that $\det A \text{ sp}_\alpha B = \det A^{1-\alpha} B^\alpha = \det A^{\frac{1-\alpha}{2}} B^\alpha A^{\frac{1-\alpha}{2}}$. Thus it suffices to prove that

$$A^{\frac{1-\alpha}{2}} B^\alpha A^{\frac{1-\alpha}{2}} \le I \implies A \text{ sp}_\alpha B \le I.$$

Since the conclusion $A \text{ sp}_\alpha B \le I$ is equivalent to $A^{\frac{1}{2}}(A \# B^{-1})^{-2\alpha} A^{\frac{1}{2}} \le I$, it is shown that, putting $\beta = -\alpha$,

$$A^{1+\beta} \le B^\beta \Longrightarrow A^{\frac{1}{2}}(A \# B^{-1})^{2\beta} A^{\frac{1}{2}} \le 1 \quad \text{for } \beta \in \left[\frac{1}{2}, 1\right].$$

Now, since $2\beta \geq 1$, it follows from (RBLP-1) that

$$\begin{aligned}
&\|A^{\frac{1}{2}}(A \# B^{-1})^{2\beta}A^{\frac{1}{2}}\| \\
&= \|A^{\frac{1}{2}}\{A^{\frac{1}{2}}(A^{-\frac{1}{2}}B^{-1}A^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}\}^{2\beta}A^{\frac{1}{2}}\| \\
&= \|A^{\frac{1}{2}}\{A^{\frac{1}{2}}CA^{\frac{1}{2}}\}^{2\beta}A^{\frac{1}{2}}\| \quad (C = (A^{-\frac{1}{2}}B^{-1}A^{-\frac{1}{2}})^{\frac{1}{2}}) \\
&\leq \|A^{\frac{1}{2}+\beta}C^{2\beta}A^{\frac{1}{2}+\beta}\| \\
&= \|A^{\frac{1}{2}+\beta}(A^{-\frac{1}{2}}B^{-1}A^{-\frac{1}{2}})^{\beta}A^{\frac{1}{2}+\beta}\| \\
&= \|A^{\beta}(A \#_{\beta} B^{-1})A^{\beta}\|.
\end{aligned}$$

We here suppose that $A^{1+\beta} \leq B^{\beta}$. Then, since $0 \leq \frac{1}{1+\beta}, \frac{2\beta}{1+\beta} \leq 1$, the Löwner-Heinz inequality ensures that $A \leq B^{\frac{\beta}{1+\beta}}$ and $A^{2\beta} \leq B^{\frac{2\beta^2}{1+\beta}}$. Hence we have

$$A^{\beta}(A \#_{\beta} B^{-1})A^{\beta} \leq A^{\beta}(B^{\frac{\beta}{1+\beta}} \#_{\beta} B^{-1})A^{\beta} = A^{\beta}B^{\frac{-2\beta^2}{1+\beta}}A^{\beta} \leq I.$$

Combining with the norm inequality obtained in above, it follows that

$$\|A^{\frac{1}{2}}(A \# B^{-1})^{2\beta}A^{\frac{1}{2}}\| \leq \|A^{\beta}(A \#_{\beta} B^{-1})A^{\beta}\| \leq 1,$$

which is the conclusion. □

Now we propose another proof in the frame of operator inequality. For this, we prepare an operator inequality version of (RBLP):

Lemma 2.2. *If $t \geq s \geq 1$, then*

$$A^{t/2}C^tA^{t/2} \leq A^{-1} \;\Rightarrow\; (A^{\frac{s}{2}}C^sA^{\frac{s}{2}})^{\frac{t}{s}} \leq A^{-1}$$

holds for $A, C > 0$.
In particular, if $t \geq 1$, then

$$A^{t/2}C^tA^{t/2} \leq A^{-1} \;\Rightarrow\; (A^{\frac{1}{2}}CA^{\frac{1}{2}})^{t} \leq A^{-1}$$

holds for $A, C > 0$.

As a matter of fact, it is obtained by noting that (RBLP) says

$$A^{\frac{1}{2}}(A^{\frac{t}{2}}C^tA^{\frac{t}{2}})A^{\frac{1}{2}} \leq I \;\Rightarrow\; A^{\frac{1}{2}}(A^{\frac{s}{2}}C^sA^{\frac{s}{2}})^{\frac{t}{s}}A^{\frac{1}{2}} \leq I$$

holds for $A, C \geq 0$ and $t \geq 1$.

Another proof of Theorem 2.1. It suffices to prove that, putting $\beta = -\alpha$,

$$A^{1+\beta} \leq B^{\beta} \Rightarrow A^{\frac{1}{2}}(A \# B^{-1})^{2\beta}A^{\frac{1}{2}} \leq 1 \quad \text{for } \beta \in \left[\frac{1}{2}, 1\right].$$

For this, we set up an operator inequality as a relay station. That is, we show that

$$A^{1+\beta} \le B^\beta \;\Rightarrow\; A^\beta(A \#_\beta B^{-1})A^\beta \le I \;\Rightarrow\; A^{\frac{1}{2}}(A \# B^{-1})^{2\beta}A^{\frac{1}{2}} \le 1.$$

Now we suppose that $A^{1+\beta} \le B^\beta$. Then the Löwner-Heinz inequality ensures that $A \le B^{\frac{\beta}{1+\beta}}$ and $A^{2\beta} \le B^{\frac{2\beta^2}{1+\beta}}$ by $0 < \frac{2\beta}{1+\beta} \le 1$. Hence we have

$$A^\beta(A \#_\beta B^{-1})A^\beta \le A^\beta(B^{\frac{\beta}{1+\beta}} \#_\beta B^{-1})A^\beta = A^\beta B^{\frac{-2\beta^2}{1+\beta}}A^\beta \le I.$$

We reach the relay station. Next we go to the goal. Putting $C = (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{-1}$, since

$$A^\beta(A \#_\beta B^{-1})A^\beta \le I \;\Leftrightarrow\; A^\beta C^\beta A^\beta \le A^{-1},$$

it follows from Lemma 2.2 for $t = 2\beta \ge 1$ that

$$A^{-1} \ge (A^{\frac{1}{2}}C^{\frac{1}{2}}A^{\frac{1}{2}})^{2\beta} = (A \# B^{-1})^{2\beta},$$

which is equivalent to the terminal. □

3 The case $-\dfrac{1}{2} \le \alpha \le 0$. The following result has been proved by Furuichi-Seo [7, Theorem 4.3]. We here give a simple proof in the frame of operator inequalities. As a matter of fact, we only use the Löwner-Heinz inequality.

Theorem 3.1. *If $-\dfrac{1}{2} \le \alpha \le 0$, then*

$$A \,\mathrm{sp}_\alpha B \prec_{\log} A^{\frac{1-\alpha}{2}}B^\alpha A^{\frac{1-\alpha}{2}}$$

holds for $A, B > 0$.

Proof. We show that if $A^{\frac{1-\alpha}{2}}B^\alpha A^{\frac{1-\alpha}{2}} \le I$, then $A^{\frac{1}{2}}(A \# B^{-1})^{-2\alpha}A^{\frac{1}{2}} \le I$. Putting $\beta = -\alpha$, we have $0 \le 2\beta \le 1 \le 1+\beta \le \frac{3}{2}$. Since $A^{1+\beta} \le B^\beta$ by the assumption, it follows that $A \le B^{\frac{\beta}{1+\beta}}$ by (LH) and so

$$(A \# B^{-1})^{-2\alpha} = (A \# B^{-1})^{2\beta} \le (B^{\frac{\beta}{1+\beta}} \# B^{-1})^{2\beta} = B^{-\frac{\beta}{1+\beta}} \le A^{-1}.$$

Hence we have

$$A^{\frac{1}{2}}(A \# B^{-1})^{-2\alpha}A^{\frac{1}{2}} \le A^{\frac{1}{2}}A^{-1}A^{\frac{1}{2}} = I,$$

as desired. □

Remark 3.2. *Bhatia, Lim and Yamazaki showed in [2, Theorem 2] that*

$$A^{\frac{1}{2}}(A \# B)A^{\frac{1}{2}} \prec_{\log} A^{\frac{3}{4}}B^{\frac{1}{2}}A^{\frac{3}{4}}$$

for $A, B > 0$ by the use of the Furuta inequality. We here remark that it is just the case $\alpha = -\frac{1}{2}$ in both Theorem 3.1 and [7, Theorem 4.3] by replacing B^{-1} by B.

Combining with Theorems 2.1 and 3.1, we showed that if $-1 \le \alpha \le 0$, then

$$A\ \mathrm{sp}_\alpha B \prec_{\log} A^{\frac{1-\alpha}{2}} B^\alpha A^{\frac{1-\alpha}{2}}$$

holds for $A, B > 0$.

Consequently we have the following result:

Theorem 3.3. *If $\alpha \in [-1, 0] \cup [1, 2]$, then*

$$A\ \mathrm{sp}_\alpha B \prec_{\log} A^{\frac{1-\alpha}{2}} B^\alpha A^{\frac{1-\alpha}{2}}$$

holds for $A, B > 0$.

Proof. Let $\alpha \in [1, 2]$. Then $\beta = 1 - \alpha \in [-1, 0]$. Hence it follows that

$$A\ \mathrm{sp}_\alpha\ B = B\ \mathrm{sp}_{1-\alpha}\ A = B\ \mathrm{sp}_\beta\ A \prec_{\log} B^{\frac{1-\beta}{2}} A^\beta B^{\frac{1-\beta}{2}} = B^{\frac{\alpha}{2}} A^{1-\alpha} B^{\frac{\alpha}{2}},$$

which is equivalent to the conclusion

$$A\ \mathrm{sp}_\alpha\ B \prec_{\log} A^{\frac{1-\alpha}{2}} B^\alpha A^{\frac{1-\alpha}{2}}.$$ □

4 An application to relative entropies.

For $0 \ne \alpha \in \mathbb{R}$, α-Tsallis relative entropies are defined as follows:

$$NT_\alpha(A||B) = -\mathrm{Tr}\ [T_\alpha(A|B)] = -\mathrm{Tr}\ \left[\frac{A\ \natural_\alpha\ B - A}{\alpha}\right], \text{ and}$$

$$D_\alpha(A||B) = -\mathrm{Tr}\ \left[\frac{A^{1-\alpha}B^\alpha - A}{\alpha}\right]$$

for $A, B > 0$. If A and B commute, then $A\ \natural_\alpha\ B = A^{1-\alpha}B^\alpha$ and hence

$$NT_\alpha(A||B) = D_\alpha(A||B)$$

We here propose a generalized relative entropy of Tsallis type which includes both $NT_\alpha(A||B)$ and $D_\alpha(A||B)$. A family of binary operations G_α among positive operators is called a generalized geometric mean if it satisfies $G_\alpha(A, B) = A^{1-\alpha}B^\alpha$ for all commuting pair $\{A, B\}$. (If necessary, we add suitable conditions.) For a family of generalized geometric mean $\{G_\alpha; 0 \ne \alpha \in \mathbb{R}\}$, a relative entropy TG_α is defined by

$$TG_\alpha(A||B) = -\mathrm{Tr}\ \left[\frac{G_\alpha(A, B) - A}{\alpha}\right] \quad \text{for } A, B > 0.$$

For convenience, we denote the transpose of the spectral geometric mean by sg_α, that is,

$$A\ \mathrm{sg}_\alpha\ B = A^{\frac{1}{2}}(A^{-1}\#B)^{2\alpha}A^{\frac{1}{2}}$$

for $A, B > 0$. Then the generalized relative entropy for it is given by

$$TSG_\alpha(A||B) = -\mathrm{Tr}\ \left[\frac{A\ \mathrm{sg}_\alpha\ B - A}{\alpha}\right] \quad \text{for } A, B > 0.$$

Thus we have the following inequality by virtue of Theorem 3.3:

Theorem 4.1. *For $-1 \leq \alpha < 0$,*

$$TSG_\alpha(A||B) \leq D_\alpha(A||B).$$

holds for $A, B > 0$.

Now we recall an important fact that Tsallis relative operator entropy $T_\alpha(A|B)$ converges to the relative operator entropy

$$S(A|B) = A^{\frac{1}{2}}(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}$$

as $\alpha \to 0$ strongly. Its spectral geometric mean version

$$TS_\alpha(A||B) = \frac{A \text{ sg}_\alpha\ B - A}{\alpha}$$

is called the α-spectral Tsallis relative operator entropy. A result corresponding to it is as in below. For this, we express an operator fidelity of A and B by

$$A\hat{\#}B = (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}.$$

Proposition 4.2. *For $A, B > 0$, a family of α-spectral Tsallis relative operator entropies $TS_\alpha(A||B)$ converges to $2S(A|A\hat{\#}B)$ as $\alpha \to 0$.*

Proof. It follows that

$$\begin{aligned} TS_\alpha(A||B) &= \frac{A \text{ sg}_\alpha\ B - A}{\alpha} \\ &= \frac{1}{\alpha}(A^{\frac{1}{2}}((A^{-1} \# B)^{2\alpha} - I)A^{\frac{1}{2}}) \\ &= 2A^{\frac{1}{2}} \cdot \frac{(A^{-1} \# B)^{2\alpha} - I}{2\alpha} \cdot A^{\frac{1}{2}} \\ &\longrightarrow 2A^{\frac{1}{2}} \log(A^{-1} \# B)A^{\frac{1}{2}} \\ &= 2A^{\frac{1}{2}}(\log A^{-\frac{1}{2}}(A\hat{\#}B)A^{-\frac{1}{2}})A^{\frac{1}{2}} \\ &= 2S(A|A\hat{\#}B), \end{aligned}$$

which completes the proof. □

Remark. We can define an α-spectral relative operator entropy of Tallis type by

$$TSP_\alpha(A||B) = \frac{A \text{ sp}_\alpha\ B - A}{\alpha} \quad \text{for } A, B > 0.$$

Unfortunately, we have not any results coresponding to Proposition 4.2. Of course, it is obvious that

$$\text{Tr } TSP_\alpha(A||B) = \text{Tr } TS_\alpha(A||B) \quad \text{for } A, B > 0.$$

It is well known that $D_\alpha(A||B) \to S_U(A||B) = \mathrm{Tr}(A\log A - \log B)$, the Umegaki relative entropy, as $\alpha \to 0$. So we have the following estimation on it by Theorem 4.1, see [7], [9], [6] and [5]. We here denote the FK-relative entropy $S_{FK}(A|B)$ for $A, B > 0$ by

$$S_{FK}(A|B) = -\mathrm{Tr}S(A|B) = -\mathrm{Tr}[A^{\frac{1}{2}}(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}].$$

Corollary 4.3. *For $A, B > 0$,*

$$2S_{FK}(A|A\hat{\#}B) \le S_U(A||B) \le S_{FK}(A|B).$$

Concluding this section, we remark log-majorization between the geometric operator mean and the operator fidelity.

Proposition 4.4. *For $A, B > 0$, $A \ \# \ B \prec_{\log} A \ \hat{\#} \ B$.*

Proof. If $A \ \hat{\#} \ B \le I$, then we have $A^{\frac{1}{2}}BA^{\frac{1}{2}} \le I$ and so $B \le A^{-1}$. It implies

$$A \ \# \ B \le A \ \# \ A^{-1} = I,$$

by which we have the conclusion. □

5 Generalized BLP inequalities. We want to cite a relation between BLP inequality and Furuta inequality. The original BLP inequality is as follows:

(BLP) For $A, B \ge 0$,

$$\|A^{\frac{1+t}{2}}B^tA^{\frac{1+t}{2}}\| \le \|A^{\frac{1}{2}}(A^{\frac{s}{2}}B^sA^{\frac{s}{2}})^{\frac{t}{s}}A^{\frac{1}{2}}\|$$

holds for $s \ge t \ge 0$.

It is known that (BLP) is led by Furuta inequality (FI). A generalization of (BLP) equivalent to (FI) is given by the following way, see [3, Th. 3.27].

(GBLP) For $A, B \ge 0$,

$$\|A^{\frac{1+t}{2}}B^tA^{\frac{1+t}{2}}\|^{\frac{p+s}{p(1+t)}} \le \|A^{\frac{1}{2}}(A^{\frac{s}{2}}B^{\frac{t(p+s)}{1+t}}A^{\frac{s}{2}})^{\frac{t}{s}}A^{\frac{1}{2}}\|$$

holds for $p \ge 1$ and $s \ge t \ge 0$.

Note that (BLP) is just the case where $p = \frac{s}{t}$ in (GBLP).
A log-majorization corresponding to (GBLP) is as follows:

Theorem 5.1. *For $A, B > 0$,*

$$(A^{\frac{1+t}{2}}B^tA^{\frac{1+t}{2}})^{\frac{p+s}{p(1+t)}} \prec_{\log} A^{\frac{1}{2}}(A^{\frac{s}{2}}B^{\frac{t(p+s)}{1+t}}A^{\frac{s}{2}})^{\frac{t}{s}}A^{\frac{1}{2}}.$$

holds for $p \ge 1$ and $s \ge t \ge 0$.

As reverses of it, we cite the following generalizations of (RBLP), cf. [3, Theorem 3.29 and 3.30] respectively:

Theorem 5.2. *For $A, B > 0$ and $\frac{1}{2} < p \leq 1$,*

$$(A^{\frac{1+s}{2}} B^{1+s} A^{\frac{1+s}{2}})^{\frac{p+s}{p(1+s)}} \succ_{\log} A^{\frac{1}{2}} (A^{\frac{s}{2}} B^{p+s} A^{\frac{s}{2}})^{\frac{1}{p}} A^{\frac{1}{2}}.$$

holds for $s \geq 0$ and $s \geq 1 - 2p$.

Theorem 5.3. *For $A, B > 0$ and $0 < p \leq \frac{1}{2}$,*

$$(A^{\frac{1+s}{2}} B^{1+s} A^{\frac{1+s}{2}})^{\frac{(2p+s)(p+s)}{p(1+s)}} \succ_{\log} A^{p+\frac{s}{2}} (A^{\frac{s}{2}} B^{p+s} A^{\frac{s}{2}})^{\frac{2p+s}{p}} A^{p+\frac{s}{2}}.$$

holds for $0 \leq s \leq 1 - 2p$.

6 Proof of (RBLP). For reader's convenience, we mention a proof of (RBLP) in this section. In the below, A, B are positive operators acting on a Hilbert space, and the Löwner-Heinz inequality is simply denoted by (LH) as in Section 2. Now we start to cite a complementary inequality of Furuta inequality due to Kamei [10]:

Theorem K. *Suppose that $A \geq B > 0$ and $0 \leq t < p \leq 1$.*
(1) *If $\frac{1}{2} \leq p \leq 1$, then $A^t \natural_{\frac{1-t}{p-t}} B^p \leq B$.*
(2) *If $0 < p \leq \frac{1}{2}$, then $A^t \natural_{\frac{2p-t}{p-t}} B^p \leq B^{2p}$.*

To prove (RBLP), we need Theorem K (1). We note that it is proved by the use of the following tools:

Lemma 6.1. *For $X, Y, Z > 0$,*
(i) $X \geq Y > 0 \Rightarrow X \natural_\alpha Z \leq Y \natural_\alpha Z$ *holds for $1 \leq \alpha \leq 2$.*
(ii) $X \natural_\alpha Y = Y X^{-1} (X \natural_{\alpha-2} Y) X^{-1} Y$.

The results in Lemma 6.1 are well-known and frequently used in many papers. Since they are basic in discussion below, we cite proofs of them, in which a basic tool is a mean theoretic version of so-called Furuta lemma:

$$X \natural_\alpha Y = Y \natural_{1-\alpha} X.$$

It is implied by $(C^*C)^\alpha = C^*(CC^*)^{\alpha-1} C$.
(i) Noting that $\beta = \alpha - 1 \in [0, 1]$ and $X^{-1} \leq Y^{-1}$, it follows from (LH) that

$$X \natural_\alpha Z = Z \natural_{1-\alpha} X = Z \natural_{-\beta} X = Z^{\frac{1}{2}} (Z^{\frac{1}{2}} X^{-1} Z^{\frac{1}{2}})^\beta Z^{\frac{1}{2}}$$

$$\leq Z^{\frac{1}{2}} (Z^{\frac{1}{2}} Y^{-1} Z^{\frac{1}{2}})^\beta Z^{\frac{1}{2}} = Y \natural_\alpha Z.$$

(ii) By the use of it, we have

$$X \natural_\alpha Y = Y \natural_{1-\alpha} X = Y(Y^{-1} \natural_{\alpha-1} X^{-1}) Y = Y(Y \natural_{\alpha-1} X)^{-1} Y.$$

Applying the obtained equation to its inside term, it follows that

$$X \natural_{\alpha} Y (= Y \natural_{1-\alpha} X) = Y[Y \natural_{\alpha-1} X]^{-1} Y$$
$$= Y[X(X \natural_{\alpha-2} Y)^{-1} X]^{-1} Y = Y[X^{-1}(X \natural_{\alpha-2} Y) X^{-1}] Y.$$

We here prepare an elementary lemma to prove Theorem K.

Lemma 6.2. *Suppose that* $\frac{1}{2} \le p \le 1$ *and* $0 \le t < p$. *If* $2 \le k \le \frac{1-t}{p-t}$, *then*

$$0 \le t \le 1 - k(p-t) \le 1 \text{ and } -1 \le -t \le 1 - k(p-t) - 2t \le 0.$$

Proof. The former is clear by $0 \le 2(p-t) \le k(p-t) \le 1-t$. To show the latter, we suppose that $2n-1 < \frac{1-t}{p-t} \le 2n$, that is,

$$0 \le t \le (2n-1)(p-t) + 1 < 1.$$

Then it follows that

$$-t = 1 - 2t - (1-t) \le 1 - 2t - k(p-t)$$
$$\le 2p - 2t - k(p-t) = (2-k)(p-t) \le 0. \qquad \square$$

Based on Lemmas 6.1 and 6.2, we give a proof to Theorem K.

Proof of Theorem K (1). First of all, we suppose that $1 < \frac{1-t}{p-t} \le 2$. Then it follows from (i) in Lemma 6.1 that

$$A^t \natural_{\frac{1-t}{p-t}} B^p \le B^t \natural_{\frac{1-t}{p-t}} B^p = B.$$

Next, if $2n-1 < \frac{1-t}{p-t} \le 2n$, then it follows from (ii) in Lemma 6.1 and Lemma 6.2 that

$$\begin{aligned}
&A^t \natural_{\frac{1-t}{p-t}} B^p \\
&= (B^p A^{-t})^{n-1} (A^t \natural_{\frac{1-t}{p-t}-2(n-1)} B^p)(A^{-t} B^p)^{n-1} \\
&\le (B^p A^{-t})^{n-1} (B^t \natural_{\frac{1-t}{p-t}-2(n-1)} B^p)(A^{-t} B^p)^{n-1} \\
&= (B^p A^{-t})^{n-1} B^{1-t-2(n-1)(p-t)+t} (A^{-t} B^p)^{n-1} \\
&\le (B^p A^{-t})^{n-1} A^{1-2(n-1)(p-t)} (A^{-t} B^p)^{n-1} \\
&= (B^p A^{-t})^{n-2} B^p A^{1-2(n-1)(p-t)-2t} B^p (A^{-t} B^p)^{n-2} \\
&\le (B^p A^{-t})^{n-2} B^p B^{1-2(n-1)(p-t)-2t} B^p (A^{-t} B^p)^{n-2} \\
&= (B^p A^{-t})^{n-2} B^{1-2(n-2)(p-t)} (A^{-t} B^p)^{n-2} \\
&\vdots \\
&= (B^p A^{-t}) B^{1-2(p-t)} (A^{-t} B^p) \\
&\le (B^p A^{-t}) A^{1-2(p-t)} (A^{-t} B^p) \\
&= B^p A^{1-2p} B^p \\
&\le B^p B^{1-2p} B^p = B.
\end{aligned}$$

On the other hand, if $2n < \frac{1-t}{p-t} \le 2n+1$, then Lemmas 6.1 (ii) and 6.2 imply that

$$\begin{aligned}
& A^t \natural_{\frac{1-t}{p-t}} B^p \\
&= (B^pA^{-t})^n (A^t \natural_{\frac{1-t}{p-t}-2n} B^p)(A^{-t}B^p)^n \\
&\le (B^pA^{-t})^n (A^t \natural_{\frac{1-t}{p-t}-2n} A^p)(A^{-t}B^p)^n \\
&= (B^pA^{-t})^n A^{1-t-2n(p-t)+t}(A^{-t}B^p)^n \\
&= (B^pA^{-t})^n A^{1-2n(p-t)}(A^{-t}B^p)^n \\
&= (B^pA^{-t})^{n-1} B^p A^{1-2n(p-t)-2t} B^p (A^{-t}B^p)^{n-1} \\
&\le (B^pA^{-t})^{n-1} B^p B^{1-2n(p-t)-2t} B^p (A^{-t}B^p)^{n-1} \\
&= (B^pA^{-t})^{n-1} B^{1-2(n-1)(p-t)}(A^{-t}B^p)^{n-1} \\
&\vdots \\
&= (B^pA^{-t}) B^{1-2(p-t)}(A^{-t}B^p) \\
&\le (B^pA^{-t}) A^{1-2(p-t)}(A^{-t}B^p) \\
&= B^pA^{1-2p}B^p \\
&\le B^pB^{1-2p}B^p = B,
\end{aligned}$$

which completes the proof. □

Finally we show a road to (RBLP) from Theorem K (1).

Proof of (RBLP). Fix $t \ge s \ge 1$. It suffices to show that

$$A^{-(1+t)} \ge B^t \;\Rightarrow\; A^{-s} \natural_{\frac{t}{s}} B^s \le A^{-(1+s)}.$$

The assumption is weakened by (LH) as $A_1 = A^{-(1+s)} \ge B^{\frac{t(1+s)}{1+t}} = B_1$. Then Theorem K (1) ensures that

$$A_1^{t_1} \natural_{\frac{1-t_1}{p_1-t_1}} B_1^{p_1} \le A_1 = A^{-(1+s)}$$

if the following conditions are satisfied; (i) $\frac{1}{2} \le p_1 \le 1$ and (ii) $0 < t_1 < p_1$. Now we take

$$t_1 = \frac{s}{1+s} \quad \text{and} \quad p_1 = \frac{s(1+t)}{t(1+s)}.$$

Then (i) and (ii) are assured by $\frac{t}{t+2} \le 1 \le s \le t$ and $p_1 - t_1 = \frac{s}{t(1+s)}$, respectively. Consequently, since $\dfrac{1-t_1}{p_1-t_1} = \dfrac{t}{s}$, it follows that

$$A^{-s} \natural_{\frac{t}{s}} B^s \le A^{-(1+s)},$$

which is equivalent to the desired conclusion by multiplying $A^{\frac{s}{2}}$ on both sides. □

In addition, we cite a proof of Theorem K (2). It is similar to that of Theorem K (1), but.

Proof of Theorem K (2). Since $\dfrac{2p-t}{p-t} = 2 + \dfrac{t}{p-t} \geq 2$, it is divided into the two cases; (i) $2n < \dfrac{2p-t}{p-t} \leq 2n+1$ for $n = 1, 2, \cdots$, and (ii) $2n-1 < \dfrac{2p-t}{p-t} \leq 2n$ for $n = 2, 3, \cdots$.

(i) We first note that under the case (i) for some n,

$$0 \leq 2k(p-t) \leq 1, \text{ and } 0 \leq 2kt - 2(k-1)p \leq 1$$

hold for $k = 1, 2, \cdots, n$. As a matter of fact, the former is ensured by

$$2k(p-t) \leq 2n(p-t) \leq 2p - t \leq 2p \leq 1,$$

and the latter is done as follows: Since $2k(p-t) \leq 2p - t$ as in above, we have $2(k-1)p < (2k-1)t < 2kt$, so that $2kt - 2(k-1)p \geq 0$, and

$$2kt - 2(k-1)p \leq 2kp - 2(k-1)p = 2p \leq 1.$$

Now it follows that

$$\begin{aligned}
& A^t \natural_{\frac{2p-t}{p-t}} B^p \\
&= (B^p A^{-t})^n (A^t \#_{\frac{2p-t}{p-t}-2n} B^p)(A^{-t} B^p)^n \\
&\leq (B^p A^{-t})^n (A^t \#_{\frac{2p-t}{p-t}-2n} A^p)(A^{-t} B^p)^n \\
&= (B^p A^{-t})^n A^{2p-t-2n(p-t)+t} (A^{-t} B^p)^n \\
&= (B^p A^{-t})^n A^{2p-2n(p-t)} (A^{-t} B^p)^n \\
&= (B^p A^{-t})^{n-1} B^p A^{2p-2n(p-t)-2t} B^p (A^{-t} B^p)^{n-1} \\
&\leq (B^p A^{-t})^{n-1} B^p B^{2p-2n(p-t)-2t} B^p (A^{-t} B^p)^{n-1} \\
&= (B^p A^{-t})^{n-1} B^{2p-2(n-1)(p-t)} (A^{-t} B^p)^{n-1} \\
&\vdots \\
&= (B^p A^{-t}) B^{2p-2(p-t)} (A^{-t} B^p) \\
&\leq (B^p A^{-t}) A^{2t} (A^{-t} B^p) \\
&= B^{2p}.
\end{aligned}$$

(ii) Next, for the case (ii) $2n-1 < \dfrac{2p-t}{p-t} \leq 2n$ for some $n \geq 2$, we note that

$0 \leq (2n-1)(p-t) \leq 2p - t \leq 2p \leq 1$, and

$0 \leq 2kt - 2(k-1)p \leq 2kp - 2(k-1)p = 2p \leq 1.$

Since $1 < \frac{2p-t}{p-t} - 2(n-1) \le 2$, it follows from Lemma 6.1 (1) that

$$
\begin{aligned}
& A^t \natural_{\frac{2p-t}{p-t}} B^p \\
&= (B^p A^{-t})^{n-1} (A^t \natural_{\frac{2p-t}{p-t}-2(n-1)} B^p)(A^{-t} B^p)^{n-1} \\
&\le (B^p A^{-t})^{n-1} (B^t \natural_{\frac{2p-t}{p-t}-2(n-1)} B^p)(A^{-t} B^p)^{n-1} \\
&= (B^p A^{-t})^{n-1} B^{2p-t-2(n-1)(p-t)+t} (A^{-t} B^p)^{n-1} \\
&\le (B^p A^{-t})^{n-1} A^{2p-2(n-1)(p-t)} (A^{-t} B^p)^{n-1} \\
&= (B^p A^{-t})^{n-2} B^p A^{2p-2(n-1)(p-t)-2t} B^p (A^{-t} B^p)^{n-2} \\
&\le (B^p A^{-t})^{n-2} B^p B^{2p-2(n-1)(p-t)-2t} B^p (A^{-t} B^p)^{n-2} \\
&= (B^p A^{-t})^{n-2} B^{2p-2(n-2)(p-t)} (A^{-t} B^p)^{n-2} \\
&\vdots \\
&= (B^p A^{-t}) B^{2p-2(p-t)} (A^{-t} B^p) \\
&\le (B^p A^{-t}) A^{2t} (A^{-t} B^p) \\
&= B^{2p}.
\end{aligned}
$$

Hence Theorem K (2) is proved. □

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